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EDITED BY

FRANK MORLEY

WITH THE COOPERATION OF

A. COHEN, CHARLOTTE A. SCOTT

AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

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To meet a generally expressed wish, and in view of the present situation, H. M. King Gustavus V has decided that the date fixed for sending in memoirs in competition for the mathematical prize founded by his Majesty be changed from the 31st October, 1916, to the 31st October, 1917.

G. MITTAG-LEFFLER.

Linear Difference and Differential Equations.

By Tomlinson Fort.

Let all the coefficients of a linear difference or differential equation have the same period ω; then if $y(x)$ is a solution of the equation $y(x+\omega)$ is also a solution. In Part I of this paper I set up equations where this fact is used as a basis for generalization. I treat equations of the second order and develop only a few fundamental facts and these primarily with their application to Part III of the paper in view although I hope that Part I will not be without interest on its own account.

Part II is minor in importance and is largely preparatory to Part III. In Part III I take up the self-adjoint boundary-value problems in one dimension. In the case of the differential equation in addition to generalization a number of new facts are brought out relative to the characteristic values for the problem. The method employed is applicable with little change to the difference equation. The results here are essentially the same as for the differential equation and so far as I know wholly new.

When a common notation does not seem practical the discussion for the difference equation is given in greater detail.

PART I.

§ 1. Definition of the Coefficients.

(a) *The Difference Equation.*

Consider the difference equation

$$L(i)y(i+1)+M(i)y(i)+N(i)y(i-1)=0, \tag{1}$$

where the independent variable i is restricted to integral values, $L(i)$, $M(i)$ and $N(i)$ defined when

$$a+1\leqq i\leqq a+\omega \tag{2}$$

and $L(i)\,N(i)\neq 0$ at any point.

Let $a_{11}(i)$ and $a_{12}(i)$ be functions defined when $a\leqq i\leqq a+\omega+1$ and subject only to the restrictions that if $a_{12}(a)\neq 0$, $a_{11}(a+1)$ and $a_{12}(a+1)$ be not both zero and that if $a_{12}(a)=0$, $a_{11}(a)\neq 0$ and $a_{11}(a+1)\neq 0$. Let $b_{11}(i)$ and $b_{12}(i)$

be functions defined when $a+\omega \leqq i \leqq a+2\omega+1$ and subject only to the restrictions that they do not both vanish at the same point and that either $b_{11}(i)$ does not vanish at all or vanishes identically. We let

$$\left.\begin{aligned} u_n(i) &= a_{11}(i)y_n(i)+a_{12}(i)y_n(i-1), \\ v_n(i) &= b_{11}(i)y_n(i+1)+b_{12}(i)y_n(i). \end{aligned}\right\} \tag{3}$$

It is desired that L, M and N be so defined, not all identically zero, when $i=a$ *and when* $a+\omega+1 \leqq i \leqq a+2\omega+1$ *that if* y_n *is any solution of* (1), *when* $a \leqq i \leqq a+\omega+1$,

$$u_n(i) \equiv v_{\bar{n}}(i+\omega), \tag{4}$$

where $y_{\bar{n}}$ *is a solution of* (1) *also.*

Let y_1 and y_2 be two linearly independent solutions of (1) and adopt the notation $(r, s;\ p, q) \equiv r(p)s(q)-r(q)s(p)$. Regard

$$u_j(i) = v_{\bar{j}}(i+\omega), \qquad j=1, 2, \tag{5}$$

as difference equations in $y_{\bar{1}}(i+\omega)$ and $y_{\bar{2}}(i+\omega)$, respectively. They are clearly solvable. We desire, moreover, that $y_{\bar{1}}(i+\omega)$ and $y_{\bar{2}}(i+\omega)$ be linearly independent. To this end define first $L(a)$, $M(a)$ and $N(a)$ so that $u_1(i)$ and $u_2(i)$ are linearly independent. This will be the case if

$$(u_1, u_2;\ a+1, a) \tag{6}$$

be different from zero.

If $a_{12}(a) \neq 0$, let a_{21} and a_{22} be two numbers such that

$$a_{11}(a+1)a_{22}-a_{12}(a+1)a_{21}=A \neq 0,$$

and let

$$\frac{M(a)}{N(a)} = \frac{a_{11}(a)-a_{22}}{a_{12}(a)}, \quad \frac{L(a)}{N(a)} = -\frac{a_{21}}{a_{12}(a)}.$$

Then using the difference equation, (6) reduces to

$$A(y_1(a+1)y_2(a)-y_2(a+1)y_1(a)),$$

which is different from zero. If $a_{12}(a)=0$, no definition of $L(a)$, $M(a)$ and $N(a)$ proves necessary.

When $b_{11} \not\equiv 0$ $y_{\bar{1}}(a+\omega)$ and $y_{\bar{2}}(a+\omega)$ are arbitrary. Let b_{21} and b_{22} be two numbers such that $b_{11}(a+\omega)b_{22}-b_{21}b_{12}(a+\omega)=B \neq 0$, and let

$$\begin{aligned} y_{\bar{1}}(a+\omega) &= b_{11}(a+\omega)u_1(a+1)-b_{21}u_1(a), \\ y_{\bar{2}}(a+\omega) &= b_{11}(a+\omega)u_2(a+1)-b_{21}u_2(a), \end{aligned}$$

whereupon by the use of equations (5)

$$\begin{aligned} \frac{M(a+\omega+1)}{L(a+\omega+1)} &= -\frac{(y_{\bar{1}}, y_{\bar{2}};\ a+\omega+2, a+\omega)}{(y_{\bar{1}}, y_{\bar{2}};\ a+\omega+1, a+\omega)} = \frac{b_{12}(a+\omega+1)-b_{21}}{b_{11}(a+\omega+1)}, \\ \frac{N(a+\omega+1)}{L(a+\omega+1)} &= \frac{(y_{\bar{1}}, y_{\bar{2}};\ a+\omega+2, a+\omega+1)}{(y_{\bar{1}}, y_{\bar{2}};\ a+\omega+1, a+\omega)} = -\frac{b_{22}}{b_{11}(a+\omega+1)}. \end{aligned}$$

Now from the difference equation $B \cdot (y_{\bar{1}},\, y_{\bar{2}};\; a+\omega+1,\; a+\omega)$ equals

$$(v_{\bar{1}},\, v_{\bar{2}};\; a+\omega+1,\; a+\omega). \tag{7}$$

But from (5), (7) equals (6) which is different from zero. Hence $(y_{\bar{1}},\, y_{\bar{2}};$ $a+\omega+1,\, a+\omega) \neq 0$, that is, $y_{\bar{1}}(i+\omega)$ and $y_{\bar{2}}(i+\omega)$ are not linearly dependent over $a \leqq i \leqq a+\omega+2$. If $b_{11}(i) \equiv 0$ the result that $(y_{\bar{1}},\, y_{\bar{2}};\; a+\omega+1,\; a+\omega)$ equals a constant different from zero times (6) and hence that $y_{\bar{1}}(i+\omega)$ and $y_{\bar{2}}(i+\omega)$ are not linearly dependent over $a \leqq i \leqq a+\omega+1$ is immediate. Moreover, no definition of $L(a+2\omega+1)$, $M(a+2\omega+1)$ and $N(a+2\omega+1)$ proves necessary.

When $i > a+\omega$, the required equation is

$$\begin{aligned}(y_{\bar{1}},\, y_{\bar{2}};\; i,\, i-1)\,y(i+1) - (y_{\bar{1}},\, y_{\bar{2}};\; i+1,\, i-1)\,y(i)&\\ + (y_{\bar{1}},\, y_{\bar{2}};\; i+1,\, i)\,y(i-1) = 0,&\end{aligned} \tag{$\bar{1}$}$$

which we adjoin to (1) as given. The coefficients are defined as desired so that if y_n is any solution there exists a solution $y_{\bar{n}}$ which satisfies (4). There is a possibility of $L(i)$ vanishing when $i > a+\omega+1$. If this should be the case we shall consider as solutions of (1) only linear combinations of $y_{\bar{1}}$ and $y_{\bar{2}}$ with constant coefficients.

(b) *The Differential Equation.*

Consider

$$L(x)y'' + M(x)y' + N(x)y = 0, \tag{1'}$$

where $L(x)$, $M(x)$ and $N(x)$ are defined and continuous when $a \leqq x \leqq a+\omega$ and accents denote differentiation, where, moreover, $\dfrac{M(x)}{L(x)}$ and $\dfrac{N(x)}{L(x)}$ are absolutely integrable over the interval $a \leqq x \leqq b$.

Let $g < a$ but $a-g$ arbitrarily small. Let $a_{11}(x)$ and $a_{12}(x)$ be absolutely integrable over $a \leqq x \leqq a+\omega$, $a_{11}(a)$ and $a_{12}(a)$ not both zero and $a'_{11}(a)$ and $a'_{12}(a)$ existent. Let $b_{11}(x)/b_{12}(x)$ be absolutely integrable when $a+\omega \leqq x \leqq a+2\omega$ and b_{11} and b_{12} both differentiable. Let, moreover,

$$\begin{aligned} u_n(x) &= a_{11}(x)\,y'_n(x) + a_{12}(x)\,y_n(x),\\ v_n(x) &= b_{11}(x)\,y'_n(x) + b_{12}(x)\,y_n(x). \end{aligned}$$

We desire that L, M and N be defined when $g \leqq x < a$ and $a+\omega \leqq x < a+2\omega$, that b_{11} and b_{12} be defined as differentiable functions when $a+\omega-g \leqq x \leqq a+\omega$ and a_{11} and a_{12} when $g \leqq x < a$ so that if y_n is any solution of (1′) there exists a solution $y_{\bar{n}}$ such that when $g \leqq x \leqq a+\omega$

$$u_n(x) = v_{\bar{n}}(x+\omega). \tag{4'}$$

Let y_1 and y_2 be linearly independent solutions of (1′) and adopt the notation $[r(x), s(x); k, l] \equiv r^{(k)}(x)\, s^{(l)}(x) - r^{(l)}(x)\, s^{(k)}(x)$ where the superscripts denote differentiation. Regard

$$u_j(x) \equiv v_j(x+\omega), \qquad j=1,2, \tag{5'}$$

as differential equations in $y_1(x+\omega)$ and $y_2(x+\omega)$ when $a \leqq x \leqq b$. They are clearly solvable. Then regard (5′) as linear algebraic equations in a_{11} and a_{12} when $g \leqq x \leqq a$ for any definition of b_{11} and b_{12} and of the coefficients. Since y_1 and y_2 are linearly independent, a_{11} and a_{12} are determined $g \leqq x \leqq a$ when $a-g$ is small enough. We desire, moreover, that $y_1(x+\omega)$ and $y_2(x+\omega)$ be linearly independent. This will be the case if $[y_1(x+\omega), y_2(x+\omega); 1, 0] \neq 0$ in the neighborhood of $x=a$. To this end we define the coefficients over $g \leqq x < a$ so that when $x \to a-0$,

$$[u_1(x), u_2(x); 1, 0] \to A[y_1(a), y_2(a); 1, 0], \tag{8}$$

where $A \neq 0$. First supposing that $a_{11}(a) \neq 0$ this is done by choosing two numbers a_{21} and a_{22} such that $a_{11}(a)a_{22} - a_{12}(a)a_{21} = A$, and then letting L, M and N be defined in such a way that as $x \to a-0$,

$$\frac{M(x)}{L(x)} \to \frac{a'_{11}(a) + a_{12}(a) - a_{21}}{a_{11}(a)} \text{ and } \frac{N(x)}{L(x)} \to \frac{a'_{12}(a) - a_{22}}{a_{11}(a)}.$$

Substitution shows that this definition is sufficient for (8). If $a_{11}(a)=0$, no definition of the coefficients in (1′) when $x<a$ will prove necessary. We let $g=a$.

We next desire that as $x \to a+\omega+0$,

$$[v_1(x+\omega), v_2(x+\omega); 1, 0] \to B[y_1(a+\omega), y_2(a+\omega), 1, 0], \tag{9}$$

where $B \neq 0$. Here if $b_{11} \not\equiv 0$ due to the arbitrariness of $y_1(a+\omega)$ and $y_2(a+\omega)$ we can proceed in a manner exactly analogous to that employed in the case of the difference equation choosing two numbers b_{21} and b_{22} such that $b_{11}(a+\omega)\, b_{22} - b_{12}(a+\omega)\, b_{21} = B$. One can then immediately assure himself, and details are omitted, that it is possible to define $b_{11}(x)$ and $b_{12}(x)$ as differentiable functions over $a+\omega-g \leqq x \leqq a+\omega$, so that if y_n is any solution of (1′), $\frac{d}{dx} v_n(x)$ is continuous at $a+\omega$, and hence so that the limit expressed by (9) is the same if the approach is from above or below. Let b_{11} and b_{12} be so defined, the desired conclusions are then drawn as under (a). If $b_{11} \equiv 0$ (9) is immediate and obvious without special definition of $y_1(a+\omega)$ and $y_2(a+\omega)$ which are no longer arbitrary.

The required equation when $g < x \leqq a+2\omega$ is

$$[y_1(x), y_2(x); 1,0]\, y'' - [y_1(x), y_2(x); 2,0]\, y' + [y_1(x), y_2(x); 2,1] y = 0. \tag{1'}$$

If $[y_1(x), y_2(x); 1, 0]$ should vanish we proceed as under (a) considering as solutions only linear combinations of y_1 and y_2 with constant coefficients.

§ 2. THE CHARACTERISTIC EQUATION.

I shall now make the substitution of the letter "x" for i of (a) of section 1 and x of (b). The reasoning applies to either the difference or the differential equation and the following notation with use of x is to include both. Other notation will be explained.

We propose the question: *Does a solution of* (1) [(1′)] *exist such that*

$$\rho u_n(x) = v_n(x+\omega)$$

where ρ *is a constant?*

Let y_1 and y_2, as previously, be two linearly independent solutions. As y_1 and y_2 are linear combinations of y_1 and y_2 we write

$$v_1(x+\omega) = \alpha_{11} u_1(x) + \alpha_{12} u_2(x),$$
$$v_2(x+\omega) = \alpha_{21} u_1(x) + \alpha_{22} u_2(x).$$

Since from the definition of the coefficients the Wronskian determinant of $v_1(x)$ and $v_2(x)$ is different from zero at $a+\omega$, $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \neq 0$.

Assume now the existence of a solution $y_3(x) = \beta_1 y_1(x) + \beta_2 y_2(x) \not\equiv 0$, such that

$$v_3(x+\omega) = \rho u_3(x).$$

Then

$$v_3(x+\omega) = \beta_1 v_1(x+\omega) + \beta_2 v_2(x+\omega) = \beta_1 \{ \alpha_{11} u_1(x) + \alpha_{12} u_2(x) \}$$
$$+ \beta_2 \{ \alpha_{21} u_1(x) + \alpha_{22} u_2(x) \} = \rho u_3(x) = \rho \beta_1 u_1(x) + \rho \beta_2 u_2(x).$$

Transposing,

$$[\beta_1(\alpha_{11}-\rho) + \beta_2 \alpha_{21}] u_1(x) + [\beta_1 \alpha_{12} + \beta_2(\alpha_{22}-\rho)] u_2(x) = 0.$$

Since $y_1(x)$ and $y_2(x)$ are linearly independent, $u_1(x)$ and $u_2(x)$ are linearly independent and hence necessarily

$$\beta_1(\alpha_{11}-\rho) + \beta_2 \alpha_{21} = 0,$$
$$\beta_1 \alpha_{12} + \beta_2(\alpha_{22}-\rho) = 0.$$

But β_1 and β_2 are not both zero as $y_3(x) \not\equiv 0$, and consequently,

$$D_1 = (\alpha_{11}-\rho)(\alpha_{22}-\rho) - \alpha_{12}\alpha_{21} = 0. \tag{10}$$

This has been derived as a necessary condition. However, if ρ is so chosen that $D_1 = 0$ we can retrace the steps and see that a solution satisfying the relation $v_n(x) \equiv \rho u_n(x)$ does exist.

Regard (10) as an equation in ρ. We shall call it the characteristic equation of (1) [(1′)]. It is of the second degree and neither root is zero as $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \neq 0$. There will always, then, exist at least one solution of (1) [(1′)], not identically zero, satisfying the relation $v_n(x+\omega) = \rho u_n(x)$.

If equation (10) has two distinct roots, ρ_1 and ρ_2, there exist at least two solutions, y_4 and y_5, not identically zero, satisfying respectively the relations

$$v_4(x+\omega)=\rho_1 u_4(x) \text{ and } v_5(x+\omega)=\rho_2 u_5(x).$$

Moreover, y_4 and y_5 are linearly independent for, assuming the contrary, there exist two constants μ_1 and μ_2 not both zero such that $\mu_1 y_4+\mu_2 y_5\equiv 0$; hence, such that

$$\mu_1 u_4(x)+\mu_2 u_5(x)\equiv 0 \quad\text{and}\quad \mu_1 v_4(x+\omega)+\mu_2 v_5(x+\omega)\equiv 0. \tag{11}$$

From the last

$$\mu_1\rho_1 u_4(x)+\mu_2\rho_2 u_5(x)\equiv 0. \tag{12}$$

For definiteness assume $\mu_1\neq 0$ and eliminate $u_5(x)$ from (11) and (12). We get

$$\mu_1(\rho_1-\rho_2)u_4(x)\equiv 0,$$

but $u_4(x)\not\equiv 0$; for it is surely possible to choose a solution y_6 linearly independent of y_4, and hence, according to section 1, such that $u_6(x)$ is linearly independent of $u_4(x)$. But this is impossible if $u_4(x)\equiv 0$. It results that $\rho_1=\rho_2$, a contradiction.

§ 3. INDEPENDENCE OF THE CHARACTERISTIC EQUATION OF THE PARTICULAR FUNDAMENTAL SYSTEM OF SOLUTIONS CHOSEN.

Consider a second fundamental system of solutions y_7 and y_8. Then

$$v_7(x+\omega)=B_{11}u_7(x)+B_{12}u_8(x),$$
$$v_8(x+\omega)=B_{21}u_7(x)+B_{22}u_8(x).$$

The characteristic equation is

$$D_2=(B_{11}-\rho)(B_{22}-\rho)-B_{12}B_{21}=0.$$

But

$$y_7(x)=L_{11}y_1(x)+L_{12}y_2(x),$$
$$y_8(x)=L_{21}y_1(x)+L_{22}y_2(x),$$

where $L_{11}L_{22}-L_{12}L_{21}\neq 0$. Hence,

$$v_7(x+\omega)=B_{11}[L_{11}u_1(x)+L_{12}u_2(x)]+B_{12}[L_{21}u_1(x)+L_{22}u_2(x)].$$

On the other hand, one can write

$$v_7(x+\omega)=L_{11}v_1(x+\omega)+L_{12}v_2(x+\omega)$$
$$=L_{11}[\alpha_{11}u_1(x)+\alpha_{12}u_2(x)]+L_{12}[\alpha_{21}u_1(x)+\alpha_{22}u_2(x)].$$

Equate and collect:

$$[(B_{11}L_{11}+B_{12}L_{21})-(L_{11}\alpha_{11}+L_{12}\alpha_{12})]u_1(x)+[(B_{11}L_{12}+B_{12}L_{22})$$
$$-(L_{11}\alpha_{12}+L_{12}\alpha_{22})]u_2(x)=0.$$

Here the coefficients must be zero as $u_1(x)$ and $u_2(x)$ are linearly independent; that is,

$$B_{11} L_{11} + B_{12} L_{21} = L_{11} \alpha_{11} + L_{12} \alpha_{21},$$
$$B_{11} L_{12} + B_{12} L_{21} = L_{11} \alpha_{12} + L_{12} \alpha_{22}.$$

Similarly,

$$B_{21} L_{11} + B_{22} L_{21} = L_{21} \alpha_{11} + L_{22} \alpha_{21},$$
$$B_{21} L_{12} + B_{22} L_{22} = L_{21} \alpha_{12} + L_{22} \alpha_{22}.$$

Hence, $D_1 = D_2$.* That is: *The characteristic equation is independent of the particular fundamental system of solutions chosen.*

§ 4. THE ROOTS OF THE CHARACTERISTIC EQUATION COINCIDENT.

Next suppose that the roots of the characteristic equation coincide. Denote this common value by ρ_1. We know that there exists at least one solution $y_4(x)$ not identically zero, such that $v_4(x+\omega) \equiv \rho_1 u_4(x)$. Let y_9 be a particular solution linearly independent of y_4, then

$$v_4(x+\omega) = \rho_1 u_4(x),$$
$$v_9(x+\omega) = c_{21} u_4(x) + c_{22} u_9(x).$$

The characteristic equation is

$$(\rho_1 - \rho)(c_{22} - \rho) = 0.$$

This must have ρ_1 a double root; hence $c_{22} = \rho_1$, that is,

$$v_9(x+\omega) = c_{21} u_4(x) + \rho_1 u_9(x), \tag{13}$$

a relation that will subsequently prove of importance.

§ 5. REAL AND IMAGINARY SOLUTIONS WHEN L, M AND N ARE REAL.

If ρ_1 and ρ_2 are distinct and y_4 and y_5 are as in section 2, one can establish the following theorems. The proofs being extremely easy are omitted.

If ρ_1 and ρ_2 are real, then y_4 and y_5 are real but for possible constant multipliers.

When ρ_1 and ρ_2 are imaginary, y_4 and y_5 are conjugate but for a possible constant multiplier.

When ρ_1 and ρ_2 are imaginary and y_4 and y_5 taken conjugate, in order for a solution $C_1 y_4 + C_2 y_5$ to be real, it is necessary and sufficient that C_1 and C_2 be conjugate.

Similarly, we can easily show that: *If ρ_1 and ρ_2 are equal they are real, and a solution satisfying the relation $v_4(x+\omega) = \rho_1 u_4(x)$ is real but for a possible constant multiplier.*

* "Theory of Differential Equations," Forsyth, Vol. IV, Part III, p. 39.

PART II.

§ 6. FUNDAMENTAL BOUNDARY-VALUE THEOREM FOR THE DIFFERENCE EQUATION.

Consider

$$\Delta\{K(i,\lambda)\,\Delta y(i)\}-G(i,\lambda)\,y(i+1)=0,^{*} \tag{14}$$

where $K(i,\lambda)>0$ is defined when $a\leqq i\leqq a+\omega$ and $G(i,\lambda)$ when $a\leqq i\leqq a+\omega-1$ for all values of the real parameter λ. Where, moreover, G is continuous in λ and such that when λ increases from $-\infty$ to ∞, it continuously increases going from $-\infty$ to ∞ also and where K is continuous in λ, never decreases and approaches a limit when $\lambda\to\infty$.

In writing G and K the argument λ will frequently be omitted. Denote an increment of λ by $\delta\lambda$ and let $\bar{\lambda}=\lambda+\delta\lambda$. Denote the corresponding functions by a $(-)$ thus $\bar{G}$, $\bar{y}$, etc. One can now establish from the difference equation the following well-known fundamental relation. Where k is any integer $a<k\leqq a+\omega$

$$\bar{K}(k)\Delta\bar{y}(k)y(k)-K(k)\Delta y(k)\bar{y}(k)-[\bar{K}(a)\Delta\bar{y}(a)y(a)-K(a)\Delta y(a)\bar{y}(a)]$$
$$=\sum_{i=a}^{k-1}(\bar{K}(i)-K(i))\Delta\bar{y}(i)\Delta y(i)+\sum_{i=a}^{k-1}(\bar{G}(i)-G(i))\bar{y}(i+1)y(i+1). \tag{15}$$

From this one concludes that if $y(a)$ and $\Delta y(a)$ are continuous in λ and are only allowed to vary so that

$$\bar{K}(a)\Delta\bar{y}(a)y(a)-K(a)\Delta y(a)\bar{y}(a)\geqq 0. \tag{16}$$

that if $\delta\lambda>0$ is sufficiently small $\bar{K}(k)\Delta\bar{y}(k)y(k)-K(k)\Delta y(k)\bar{y}(k)>0$, that is, that $\dfrac{K(k)\Delta y(k)}{y(k)}$ increases with λ when defined and that since when $y(k)=0$ and $|\delta\lambda|>0$ sufficiently small $-K(k)\Delta y(k)\bar{y}(k)\neq 0$ roots of $y(k)$ do not cluster. One can now conclude without difficulty that nodes† of y move continuously to the right as λ increases.

We now define y and K as continuous functions; namely, those functions defined by the broken line graphs of $y(i)$ and $K(i)$, respectively. Let y' denote the forward derivative of y. Let, moreover, $\bar{x}$ be a fixed point, $a\leqq\bar{x}<a+1$. When if $c_1y'(\bar{x})+c_2y(\bar{x})\equiv 0$ where c_1 and c_2 are independent of λ, the assumption that y does not satisfy (16) leads to an immediate contradiction. Hence, if when λ varies, we require that $c_1y'(\bar{x})+c_2y(\bar{x})=0$, we thereby require that (16) be satisfied.

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* An equation $L(i)\,y(i+1)+M(i)\,y(i)+N(i)\,y(i-1)=0$ where $L(i)\,M(i)>0$, can be written in the form $\Delta\{K(i)\,\Delta y(i)\}-G(i)\,y(i+1)=0$ where $K(i)>0$; and, conversely.

† A real function defined only at integral points can be plotted in the ordinary Cartesian plane as a succession of isolated points. Join these points with straight line segments. The whole will, in general, form a broken line which will be spoken of as the broken line graph of the function. The points where it crosses the axis are called nodes.

One can now conclude in various ways* that when $\lambda > 0$ is sufficiently large a solution satisfying $c_1 y'(\bar{x}) + c_2 y(\bar{x}) = 0$ has no node on $a+1 < x \leqq a+\omega+1$ and that $\frac{\Delta y(a+\omega)}{y(a+\omega)}$ is positive and as large as we wish to make it, and that, moreover, when $\lambda < 0$ is numerically sufficiently large not only are there the maximum number of nodes on $a+1 < x \leqq a+\omega+1$ but that the last is as close to $a+\omega$ as we desire, that is, $\frac{\Delta y(a+\omega)}{y(a+\omega)}$ is negative and numerically as large as we wish to have it.

Moreover, the requirement that $y(\bar{x}+\omega) = 0$ is equivalent to a requirement of the form $\bar{c}_1 K(a+\omega) \Delta y(a+\omega) + \bar{c}_2 y(a+\omega) = 0$. We, consequently, conclude:

Let y be a solution satisfying $c_1 y'(\bar{x}) + c_2 y(\bar{x}) = 0$. There exist values of λ, $\lambda_0 > \lambda_1 > \ldots\ldots > \lambda_k$ such that when $\lambda = \lambda_j$, $j = 0, \ldots\ldots, k$, $y(\bar{x}+\omega) = 0$ also; where, moreover, y has exactly j nodes on the interval $a+1 < x \leqq a+\omega$. If $y(a+1) \neq 0$, $k = \omega - 1$, if $y(a+1) = 0$, $k = \omega - 2$.

§ 7. Extension of Fundamental Boundary-Value Theorem to Equations of the Type Set Up in Section 1.

(a) *The Difference Equation.*

Write equation (14) in the form

$$L(i, \lambda)\, y(i+1) + M(i, \lambda)\, y(i) + N(i, \lambda)\, y(i-1) = 0. \tag{17}$$

Here $L(i, \lambda)$, $M(i, \lambda)$ are defined for all values of λ when $a+1 \leqq i \leqq a+\omega$. We let $u_n(i) = y_n(i-1)$, $v_n(i+\omega) = d_{11}(i)\, K(a+\omega)\, \Delta y_n(i+\omega) + d_{12}(i)\, y_n(i+\omega)$, where $d_{11}(i)$ and $d_{12}(i)$ are real functions defined when $a \leqq i \leqq a+\omega+1$, $d_{11}(i) \neq 0$ or $d_{11} \equiv 0$. We then define L, M and N according to the method of section 1 so that if y_n is any solution there exists a solution $y_{\bar{n}}$ such that $u_n(i) = v_{\bar{n}}(i+\omega)$. *$u_n$ and $v_{\bar{n}}$ are to be considered functions of the continuous variable x, namely, those functions defined by the broken line graphs of $u_n(i)$ and $v_n(i)$, respectively.* We write moreover in place of the $a_{21} y_n(a+1) + a_{22} y_n(a)$ of section 1 $K(a)\, \Delta y_n(a) + b y_n(a) = \bar{u}_n(a+1)$ and in place of the $b_{21} y_n(a+\omega+1) + b_{22} y_n(a+\omega)$ of section 1 $d_{21} K(a+\omega)\, \Delta y_n(a+\omega) + d_{22} y_n(a+\omega) = \bar{v}_n(a+\omega)$ where $d_{11} d_{22} - d_{12} d_{21} = 1$.

Let $a+1 \leqq \bar{x} \leqq a+\omega+1$ and let y_n denote a particular solution, not identically zero, such that $u_n(\bar{x}) = 0$. Denote by c that integer such that $c < \bar{x} \leqq c+1$. Form the equation

$$\bar{L}(i, \lambda)\, y(i+1) + \bar{M}(i, \lambda)\, y(i) + \bar{N}(i, \lambda)\, y(i-1) = 0, \tag{18}$$

* A satisfactory method of proving this theorem is by comparison with $\zeta \Delta^2 y(i) - P(i)\, y(i+1) = 0$ where P is a constant chosen large numerically and positive or negative as we desire, and ζ is a constant such that $0 < \zeta < K(i)$.

where $\bar{L}(i,\lambda)$, $\bar{M}(i,\lambda)$ and $\bar{N}(i,\lambda)$ are equal respectively to $L(i,\lambda)$, $M(i,\lambda)$ and $N(i,\lambda)$ when $c+1\leqq i\leqq a+\omega$ and to $L(i-\omega-1,\lambda)$, $M(i-\omega-1,\lambda)$ and $N(i-\omega-1,\lambda)$ respectively when $a+\omega+1\leqq i\leqq c+\omega+1$.

Now denote by y_{10} that solution of (14) such that $v_n(i+\omega)=u_{10}(i)$. Then there exists a solution of (18) which we denote by y_{11} such that $y_{11}(i)=y_{10}(i-\omega-1)$ over $a+\omega\leqq i\leqq c+\omega+2$. Therefore, $u_{11}(a+\omega+2)=v_n(a+\omega+1)$ and $u_{11}(a+\omega+1)=v_n(a+\omega)$. Due to the definition given $L(a,\lambda)$, $M(a,\lambda)$, $N(a,\lambda)$, $L(a+\omega+1,\lambda)$, $M(a+\omega+1,\lambda)$ and $N(a+\omega+1,\lambda)$, these equations reduce to

$$\left.\begin{aligned}\bar{u}_{11}(a+\omega+2)&=v_n(a+\omega),\\ u_{11}(a+\omega+2)&=\bar{v}_n(a+\omega).\end{aligned}\right\}\qquad(20)$$

Hence since y_n satisfies a relation like (16) at $a+\omega$, y_{11} satisfies the same relation at $a+\omega+1$.

Solving (20) $y_{11}(a+\omega+2)=\bar{D}K(a+\omega)\Delta y_n(a+\omega)+Dy_n(a+\omega)$. We shall say that we have case α if $\bar{D}\neq 0$ and case β if $\bar{D}=0$. Apply the theorem of oscillation for y_n and we have that as λ increases from $-\infty$ to ∞ $y_{11}(a+\omega+2)$ vanishes exactly m times where

$$\begin{aligned}&\text{CASE }\alpha\text{:}\quad &&\text{(a)}\ y_n(c+1)\neq 0,\ m=a+\omega-c,\\ & &&\text{(b)}\ y_n(c+1)=0,\ m=a+\omega-c-1;\\ &\text{CASE }\beta\text{:} &&\text{(a)}\ y_n(c+1)\neq 0,\ m=a+\omega-c-1,\\ & &&\text{(b)}\ y_n(c+1)=0,\ m=a+\omega-c-2.\end{aligned}$$

I shall next show that: *If λ is positive and sufficiently large y_{11} has no node on $a+\omega+2<x\leqq\bar{x}+\omega$ and if λ is negative and numerically sufficiently large, the maximum number namely, $c-a-2$.*

CASE α. Here $\dfrac{\Delta y_{11}(a+\omega+1)}{y_{11}(a+\omega+2)}$ approaches a limit as λ becomes infinite and one can immediately prove the theorem, as for example, as indicated in the third footnote under section 6.

CASE β. First, when λ is positive and sufficiently large a solution of (18) exists having no node* on $a+\omega+1\leqq x\leqq c+\omega+1$. Hence as the nodes of two solutions either separate each other or coincide, no solution can have more than one node on this interval. But under these circumstances, as $y_{11}(a+\omega+2)$ is proportional to $y_n(a+\omega)$, $y_{11}(a+\omega+1)$ which is also a linear combination of $K(a+\omega)\Delta y_n(a+\omega)$ and $y_n(a+\omega)$ can not be proportional to $y_n(a+\omega)$ also consistent with the equation $d_{11}(a)d_{22}-d_{21}d_{12}(a)=1$; and hence $y_{11}(a+\omega+1)$ vanishes one more time than $y_{11}(a+\omega+2)$ as λ increases from $-\infty$ to ∞, and

* This will be the true of any solution having the same sign at $a+\omega+1$ and $a+\omega+2$.

as when $x \geqq a+\omega+1$ its nodes move to the left as λ decreases and to the right as λ increases when $\lambda=\infty$ there must remain one node of y_{11} on the interval $a+\omega+1<x<a+\omega+2$, and hence when λ is sufficiently large y_{11} can have no node on $a+\omega+2<x\leqq c+\omega+1$.

Next, when λ is negative and numerically sufficiently large a solution exists having the maximum number of nodes* on $a+\omega+1\leqq x\leqq c+\omega+1$ conceivably possible, and hence y_{11} surely has on this interval at least as many nodes as one less than this maximum number. But as λ decreases from ∞ to $-\infty$ $y_{11}(a+\omega+1)$ vanishes one more time than $y_{11}(a+\omega+2)$, and hence when λ is negative and numerically sufficiently large y_{11} has no node on the interval $a+\omega+1\leqq x\leqq a+\omega+2$ and hence the maximum number of nodes on $a+\omega+2 <x\leqq c+\omega+1$.

Now bearing in mind that as λ decreases nodes of y_{11} on the interval $a+\omega+1\leqq x\leqq c+\omega+1$ move continuously to the left, that

$$\frac{K(a+\omega)\,\Delta y_{11}(a+\omega)}{y_{11}(a+\omega)} \to -\infty$$

as $\lambda \to -\infty$ and that $y_{11}(\bar{x}+\omega)=u_{11}(\bar{x}+\omega+1)=u_{11}(\bar{x})=v_n(\bar{x}+\omega)$ a mere count gives the following theorem:

There exist values of λ, $\lambda_0>\lambda_1>\ldots.>\lambda_k$ *such that when* $\lambda=\lambda_j$, $j=0, \ldots., k$, *each solution of* (14) *satisfying the relation* $y_n(\bar{x}-1)=0$ *satisfies also the relation* $v_n(\bar{x}+\omega)=0$.

CASE α: (a) $k=\omega-1$,
(b) $k=\omega-2$;

CASE β: (a) $k=\omega-2$,
(b) $k=\omega-3$.

The notation "λ_j" for any particular one of the values $\lambda_0, \ldots., \lambda_k$ *will be generally used.*

(b) *The Differential Equation.*

We consider the equation

$$\frac{d}{dx}\left(K(x,\lambda)\,y'(x)\right)-G(x,\lambda)\,y(x)=0,$$

where the prime denotes differentiation and where K and G are defined and continuous for all values of λ when $a\leqq x\leqq a+\omega$ and where moreover $G(x,\lambda)$ is continuous in λ and continually increases going from $-\infty$ to ∞ as λ increases from $-\infty$ to ∞ and where $K>0$ never decreases.

* This will be the true of any solution having opposite signs at $a+\omega+1$ and $a+\omega+2$.

We write this equation in the form

$$L(x,\lambda)y''+M(x,\lambda)y'+N(x,\lambda)y=0$$

and define L, M and N over the interval $a+\omega<x\leqq a+2\omega$, according to the method of section 1, so that if y_n is any solution there exists a solution $y_{\bar{n}}$ such that $u_n(x)\equiv v_{\bar{n}}(x+\omega)$ where

$$u_n(x)=y_n(x) \quad\text{and}\quad v_{\bar{n}}(x+\omega)=d_{11}(x)K(a+\omega)y'_{\bar{n}}(x+\omega)+d_{12}(x)y_{\bar{n}}(x+\omega)$$

where d_{11} and d_{12} are given functions as in section 1 (b). The analogue of (a) of this section is then immediate. Form the equation

$$\bar{L}(x,\lambda)y''+\bar{M}(x,\lambda)y'+\bar{N}(x,\lambda)y=0, \tag{18'}$$

where $\bar{L}=L$, $\bar{M}=M$ and $\bar{N}=N$ when $\bar{x}\leqq x\leqq a+\omega$ and to $L(x-\omega,\lambda)$ etc., when $a+\omega\leqq x\leqq\bar{x}+\omega$ and reason as under (a).

There exist an infinite number of real values of λ, $\lambda_0>\lambda_1>\lambda_2>\ldots$, *such that when* $\lambda=\lambda_j$, $j=0,1,2,\ldots$, *all solutions satisfying* $u_n(\bar{x})=0$ *satisfy also* $v_n(\bar{x}+\omega)=0$; $\bar{x}$ *any particular point* $a\leqq x\leqq a+\omega$.

§ 8. CONTINUITY PROOF.

(a) *The Difference Equation.*

It is apparent that the λ_j's of section 7 will, in general, be functions of the point $\bar{x}$. We write $\lambda_j(\bar{x})$ and *shall prove that these functions, with the exception of the last* λ_k, *are continuous* $a+1\leqq\bar{x}\leqq a+\omega+1$ *and that* λ_k *is continuous over every interval,* $i<\bar{x}<i+1$, *where* i *is an integer becoming negatively infinite as* $\bar{x}$ *approaches* i *or* $i+1$.

Let ξ be any particular fixed value of $\bar{x}$ and let y_ξ be a particular solution, not identically zero, such that $u_\xi(\xi)=y_\xi(\xi-1)=0$. Let $\delta>0$ and let $0<\xi-\bar{x}<\delta$; and then let $y_{\bar{x}}$ be a solution satisfying the conditions $y_{\bar{x}}(\bar{x}-1)=0$ and $y'_{\bar{x}}(\bar{x}-1)=y'_\xi(\xi-1)$ where as previously the accent denotes the forward derivative. Let c be that integer such that $c<\xi-1\leqq c+1$ and let $\varepsilon>0$ be arbitrarily small. Then, if δ is small enough $|y_{\bar{x}}(c)-y_\xi(c)|<\varepsilon$ and $|y_{\bar{x}}(c+1)-y_\xi(c+1)|<\varepsilon$. But $y_\xi(a+\omega)$ and $y_\xi(a+\omega+1)$ are polynomials in $y_\xi(c)$ and $y_\xi(c+1)$; and moreover, $y_{\bar{x}}(a+\omega)$ and $y_{\bar{x}}(a+\omega+1)$ are the same polynomials in $y_{\bar{x}}(c)$ and $y_{\bar{x}}(c+1)$. As polynomials are continuous, if ε be sufficiently small $y_{\bar{x}}(a+\omega)$ and $y_{\bar{x}}(a+\omega+1)$ can be made to differ in absolute value as little as we please from $y_\xi(a+\omega)$ and $y_\xi(a+\omega+1)$, respectively.

Now consider equation (18) formed for c and the solutions of it $\bar{\bar{y}}_\xi$ and $\bar{\bar{y}}_{\bar{x}}$, bearing the relations to y_ξ and $y_{\bar{x}}$, respectively, expressed by (20) and hence where $\eta>0$ is arbitrary, if ε is small enough $|\bar{\bar{y}}_\xi(a+\omega+1)-\bar{\bar{y}}_{\bar{x}}(a+\omega+1)|<\eta$

and $|\bar{\bar{y}}_\xi(a+\omega+2)-\bar{\bar{y}}_{\bar{x}}(a+\omega+2)|<\eta$. Moreover, for larger values of i $\bar{\bar{y}}_\xi$ and $\bar{\bar{y}}_x$ are polynomials in these values.

Let $\lambda=\lambda_j(\xi)$, $j<k$ if $\xi=c+1$, otherwise $j\leqq k$; then a node which we call the $(j+1)$-st node of $\bar{\bar{y}}_\xi$ lies at $\xi+\omega$. Let $\theta>0$, then if η be small enough, the $(j+1)$-st node of $\bar{\bar{y}}_x$ lies a distance from $\bar{x}+\omega$ less than θ. Now hold $\bar{x}$ fast and vary λ. As nodes of $\bar{\bar{y}}_{\bar{x}}$ move continuously to the right as λ increases, and to the left as λ decreases; by a variation in λ as small as we please, θ sufficiently small, the $(j+1)$-st node of $\bar{\bar{y}}_{\bar{x}}$ can be made to move to $\bar{x}+\omega$. When it will have moved to this point, λ will have the value $\lambda_j(\bar{x})$. That is given a $\zeta>0$ we can choose a θ, then an η, then an ϵ, then a δ so that $|\lambda_j(\bar{x})-\lambda_j(\xi)|<\zeta$ when $0<\xi-\bar{x}<\delta$.

Now let $0\leqq\bar{x}-\xi<\bar{\delta}$ and let c be that integer such that $c\leqq\xi-1<c+1$. We form equation (18) for this point c. The corresponding theory developed with the relations between the y's and $\bar{\bar{y}}$'s now holds without the slightest change. In fact, it was quite immaterial whether we chose c that integer $c<\bar{x}-1\leqq c+1$, as we did or $c\leqq\bar{x}-1<c+1$. The repetition of the reasoning just gone through now shows that given a $\zeta>0$ it is possible to choose a $\bar{\delta}>0$ so that when $\bar{x}-\xi<\bar{\delta}$, $|\lambda_j(\bar{x})-\lambda_j(\xi)|<\zeta$.

Let $\bar{\bar{\delta}}\leqq\delta$, $\bar{\delta}$ then given a $\zeta>0$ it is possible to choose a $\bar{\bar{\delta}}>0$ such that when $|\bar{x}-\xi|<\bar{\bar{\delta}}$, $|\lambda_j(\bar{x})-\lambda_j(\xi)|<\zeta$. This establishes continuity.

The exceptional situation in the case of λ_k arises from the fact that if $\bar{x}-1$ is integral, there exists no value λ_k. We have case α (b) or case β (b). Let $\bar{x}-1$ approach an integer $c+1$ from below. Consider (18) formed for c. The value of λ which makes $\bar{\bar{y}}(a+\omega+2)$ vanish the $(a+\omega-c)$-th or the $(a+\omega-c-1)$-st time, as we have case α (b) or case β (b), becomes negatively infinite; but at all times λ is less than or equal to this value and hence itself becomes negatively infinite. Similarly, if $\bar{x}-1$ approaches an integer c from above we think of (18) where c is that integer $c\leqq\bar{x}-1<c+1$ with exactly the same results as when $\bar{x}-1$ approaches $c+1$ from below.

(b) *The Differential Equation.*

All the values of $\lambda_j(\bar{x})$ *are continuous* $a\leqq\bar{x}\leqq a+\omega$ *in the case of the differential equation.*

Here we make use of a few fundamental and well-known theorems for the differential equation and follow the general method employed in (a). There is less need for detail; and naturally no supplementary discussion for a particular λ_j is necessary as for the λ_k under (a). I shall again for the sake of brevity omit details of the proof.

PART III.

§ 9. STATEMENT OF PROBLEM.

(a) *The Difference Equation.*

Consider again

$$\Delta\{K(i,\lambda)\,\Delta y(i)\}-G(i,\lambda)\,y(i+1)=0, \tag{21}$$

where $K(i,\lambda)>0$ and $G(i,\lambda)$ are defined when

$$a\leqq i\leqq a+\omega \tag{22}$$

and

$$a\leqq i\leqq a+\omega-1 \tag{23}$$

respectively, where moreover K and G are continuous in the real parameter λ, when $-\infty<\lambda<\infty$ and such that as λ increases G always actually increases going from $-\infty$ to ∞ and K does not decrease approaching a limit as $\lambda\to\infty$.

We shall consider (21) subject to the conditions

$$\left.\begin{aligned}\alpha_{11}K(a)\,\Delta y(a)+\alpha_{12}y(a)&=\beta_{11}K(a+\omega)\,\Delta y(a+\omega)+\beta_{12}y(a+\omega),\\ \alpha_{21}K(a)\,\Delta y(a)+\alpha_{22}y(a)&=\beta_{21}K(a+\omega)\,\Delta y(a+\omega)+\beta_{22}y(a+\omega),\end{aligned}\right\} \tag{24}$$

where $\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}=\beta_{11}\beta_{22}-\beta_{21}\beta_{12}$.

It proves advisable to throw conditions (24) into the form

$$\left.\begin{aligned}K(a)\,\Delta y(a)+b\,y(a)&=d_{11}K(a+\omega)\Delta y(a+\omega)+d_{12}y(a+\omega),\\ y(a)&=d_{21}K(a+\omega)\Delta y(a+\omega)+d_{22}y(a+\omega),\end{aligned}\right\} \tag{25}$$

where $d_{11}\neq 0$, $d_{11}d_{22}-d_{21}d_{12}=1$.

It will prove advantageous also to consider side by side with (25)

$$\left.\begin{aligned}K(a)\,\Delta y(a)+by(a)&=-d_{11}K(a+\omega)\,\Delta y(a+\omega)-d_{12}y(a+\omega),\\ y(a)&=-d_{21}K(a+\omega)\,\Delta y(a+\omega)-d_{22}y(a+\omega).\end{aligned}\right\} \tag{26}$$

Let $u_n(i)=y_n(i-1)$ and $v_n(i+\omega)=d_{11}K(a+\omega)\,\Delta y_n(i+\omega)+d_{12}y_n(i+\omega)$. We then define the coefficients as in section 1 so that if y_n is any solution of (21) there exists a solution $y_{\bar n}$ such that $u_n(i)\equiv v_{\bar n}(i+\omega)$ and also $u_n(a)=K(a)\,\Delta y_n(a)+b\,y_n(a)=\bar u_n(a+1)$ and $v_n(a+\omega+1)=d_{21}K(a+\omega)\,\Delta y_n(a+\omega)+d_{22}y_n(a+\omega)=\bar v_n(a+\omega)$. It is only necessary to replace the a_{21}, a_{22}, b_{21} and b_{22} of section 1 by $K(a)$, $-K(a)+b$, $d_{21}K(a+\omega)$ and $-d_{21}K(a+\omega)+d_{22}$, respectively, and proceed as there.

The existence of a solution of (21) *not identically zero satisfying* (25) *now becomes necessary and sufficient for the existence of a solution not identically zero satisfying identically the relation*

$$u_n(i)=v_n(i+\omega),$$

and the existence of a solution not identically zero satisfying (26) *becomes necessary and sufficient for the existence of a solution not identically zero satisfying identically the relation*

$$u_n(i)=-v_n(i+\omega).$$

The sufficiency of these conditions is immediate. The necessity is proven thus. Suppose that there exists a solution satisfying (25). Denote this by y_1. Then

$$K(a)\,\Delta y_1(a)+b\,y_1(a)=d_{11}K(a+\omega)\,\Delta y_1(a+\omega)+d_{12}y_1(a+\omega)$$
$$y_1(a)=d_{21}K(a+\omega)\,\Delta y_1(a+\omega)+d_{22}y_1(a+\omega).$$

We know that there exists a solution $y_{\bar{1}}$ such that $u_1(i)=v_{\bar{1}}(i+\omega)$. Then

$$K(a)\,\Delta y_1(a)+b\,y_1(a)=d_{11}K(a+\omega)\,\Delta y_{\bar{1}}(a+\omega)+d_{12}y_{\bar{1}}(a+\omega)$$
$$y_1(a)=d_{21}K(a+\omega)\,\Delta y_{\bar{1}}(a+\omega)+d_{22}y_{\bar{1}}(a+\omega),$$

and consequently, $y_{\bar{1}}$ is identical with y_1 at $a+\omega$ and $a+\omega+1$ and hence at all points.

We now define $u_n(i)$ and $v_n(i)$ as functions of the continuous variable x, as has previously been done; namely, as those functions defined by the broken line graphs of $u_n(i)$ and $v_n(i)$, respectively. It results that the satisfaction of $u_n(x)\equiv v_n(x+\omega)$ by a solution of (21), not identically zero, is necessary and sufficient for the satisfaction of (25) by such a solution, and the satisfaction of $u_n(x)\equiv -v_n(x+\omega)$ is necessary and sufficient for the satisfaction of (26).

If values of λ exist such that (21) and (25) are simultaneously satisfied by a function of i, not identically zero, we shall speak of them as values l; and if values exist so that (21) and (26) are simultaneously satisfied by a function of i, not identically zero as values of l'. The solutions of (21) not identically zero satisfying (25) or (26) as the case may be will be called corresponding solutions.

(b) *The Differential Equation.*

We consider here again

$$\frac{d}{dx}\{K(x,\lambda)\,y'(x)\}-G(x,\lambda)\,y(x)=0,$$

where $K(x,\lambda)>0$ and $G(x,\lambda)$ for all values of the real parameter λ are defined and continuous when

$$a\leqq x\leqq a+\omega, \tag{22'}$$

and where K and G are continuous in λ, and such that as λ increases G increases, constantly going from $-\infty$ to ∞ and K does not ever decrease.

We consider (21′) subject to

$$\left.\begin{aligned}K(a)\,y'(a)&=d_{11}K(a+\omega)\,y'(a+\omega)+d_{12}y(a+\omega)\\ y(a)&=d_{21}K(a+\omega)\,y'(a+\omega)+d_{22}y(a+\omega),\end{aligned}\right\}\tag{25'}$$

where $d_{11}d_{22}-d_{12}d_{21}=1$ and also side by side with these subject to

$$\left.\begin{aligned}K(a)\,y'(a)&=-d_{11}K(a+\omega)\,y'(a+\omega)-d_{12}y(a+\omega)\\ y(a)&=-d_{21}K(a+\omega)\,y'(a+\omega)-d_{22}y(a+\omega).\end{aligned}\right\}\tag{26'}$$

Suppose $d_{21} \neq 0$.* Let $u_n(x) = y_n(x)$, $v_n(x+\omega) = d_{21} K(a+\omega) \Delta y_n(i+\omega) + d_{22} y(i+\omega)$.

We proceed in a manner similar to that employed under (a) of this section, defining the coefficients of the differential equation over $g \leq x < a$ and $a+\omega < x \leq a+2\omega$, etc., according to section 1 so that the identical satisfaction of the relation $u_n(x) = v_n(x+\omega)$ by a solution of (21′) not identically zero is necessary and sufficient for the satisfaction of (25′) by the same solution. The identical satisfaction of $u_n(x) = -v_n(x+\omega)$ is then also necessary and sufficient for the satisfaction of (26′) by the same solution. We let

$$\bar{u}_n(a) = y'_n(a) \text{ and } \bar{v}_n(a+\omega) = \frac{1}{K(a)} [d_{11} K(a+\omega) y'(a+\omega) + d_{12} y(a+\omega)].$$

§ 10. TO PROVE THAT THE MAXIMA† AND MINIMA OF THE VALUES λ_j ARE VALUES l OR l'.

Under the term maximum (minimum) we include the case that the function is a constant or has a constant value over a neighborhood.

The constant term of the characteristic equation is unity since A and B of section 1 are equal.

Suppose $\lambda_j(\bar{a})$ *a maximum. To prove that when* $\lambda = \lambda_j(\bar{a})$ $\rho = \pm 1$.

Assume that this were not the case then $\rho_1 \neq \rho_2$ as $\rho_1 \rho_2 = 1$.

Denote by y_1 and y_2 two solutions, neither identically zero, such that $\rho_1 u_1(x) = v_1(x+\omega)$ and $\rho_2 u_2(x) = v_2(x+\omega)$. Then

$$u_1(x) v_2(x+\omega) - u_2(x) v_1(x+\omega) = (\rho_1 - \rho_2) u_1(x) \cdot u_2(x).$$

Let $y_n(x)$ be a solution such that $u_n(\bar{a}) = v_n(\bar{a}+\omega) = 0$, $y_n \not\equiv 0$; in the case of the difference equation $a+1 \leq \bar{a}$ and in the case of the differential equation $a \leq \bar{a}$. Suppose $y_n = c_1 y_1 + c_2 y_2$, then

$$0 = c_1 u_1(a) + c_2 u_2(a),$$
$$0 = c_1 \rho_1 u_1(a) + c_2 \rho_2 u_2(a).$$

As c_1 and c_2 are not both zero $(\rho_1 - \rho_2) u_1(\bar{a}) \cdot u_2(\bar{a}) = 0$. But $\rho_1 \neq \rho_2$ and hence $u_1(\bar{a}) \cdot u_2(\bar{a}) = 0$. They are not both zero and excluding $\bar{a}$ itself in some neighborhood of $\bar{a}$ neither is zero.‡ Suppose for definiteness $u_1(\bar{a}) = 0$; then y_1 is

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* In case $d_{21} = 0$ we simply let $v_n(x+\omega) = D_{21}(x) y'_n(x+\omega) + D_{22}(x) y_n(x+\omega)$, where $D_{21}(a+\omega) = 0$, $D_{22}(a+\omega) = d_{22}$, $D'_{21}(a+\omega) + D_{22}(a+\omega) = \frac{d_{11}}{K(a)}$ and $D'_{22}(a+\omega) = \frac{d_{12}}{K(a)}$. Then let y_{I} and y_{II} be so chosen, first and second derivatives existent, that, if $D_{21}(x) y'_{\text{I}}(x+\omega) + D_{22}(x) \bar{y}_1 = y_1$, $D_{22}(x) y'_{\text{II}}(x-\omega) + D_{22}(x) y_{\text{II}} = y_2$, D_{21} and D_{22} together with their derivatives at $a+\omega$ take on values as stated and are existent at other points. This can be done as is seen by solving these equations for D_{21} and D_{22}. Having made one determination of D_{21} and D_{22} regard them as fixed. The problem now is essentially the same as if $d_{21} \neq 0$.

† The existence of extremes will be discussed in section 14.

‡ In the case of the difference equation $u_n(x) = y_n(x-1)$ and in the case of the differential equation $u_n(x) = y_n(x)$.

essentially real, for write $y_1 \equiv y_j + y_k\sqrt{-1}$, where y_j and y_k are real. Then $u_j(\bar{a}) = u_k(\bar{a}) = 0$ and hence y_j and y_k are proportional. Consequently, $y_1 \equiv c y_j$. Let $y_1 \equiv y_j$. It results that ρ_1 is real, whence ρ_2 is also. Hence y_2 is essentially real and will be considered real. $u_1(x)$ changes sign at $\bar{a}$, and hence the determinant (27) changes sign at $\bar{a}$ also.

Let y_3 and y_4 be any two real linearly independent solutions.

$$\left.\begin{aligned} y_3 &= k_{11} y_1 + k_{12} y_2, \\ y_4 &= k_{21} y_1 + k_{22} y_2. \end{aligned}\right\} \qquad (28)$$

$$\left.\begin{aligned} \Delta(x) &\equiv u_3(x) \cdot v_4(x+\omega) - u_4(x) \cdot v_3(x+\omega) \\ &= (k_{11} k_{22} - k_{12} k_{21})\,(u_1(x) v_2(x+\omega) - u_2(x) v_1(x+\omega)). \end{aligned}\right\} \qquad (29)$$

It results that $\Delta(x)$ changes sign at $\bar{a}$ and that if $0 < \delta' < \epsilon$, sufficiently small, $\Delta(\bar{a}+\delta') \cdot \Delta(\bar{a}-\delta') < 0$.

Under the assumption that $\lambda_j(\bar{a})$ is a maximum, I shall derive a contradiction to this statement.

Denote by y_5 and y_6, respectively, solutions such that $u_5(\bar{a}-\delta') = 0$ and $u_6(\bar{a}+\delta') = 0$ and such that at $\bar{a}-\delta'$ and $\bar{a}+\delta'$, respectively, the forward slopes of u_5 and u_6 are the same as that of u_1 at $\bar{a}$.

In the case of the differential equation let $c = \bar{a}$ and in the case of the difference equation let c be that integer, $c < \bar{a} - 1 \leqq c + 1$. Form equation (18) [(18′)] of section 7 for this point c. There exist solutions $\bar{\bar{y}}_1$, $\bar{\bar{y}}_5$ and $\bar{\bar{y}}_6$ such that

$$\begin{aligned} v_1(\bar{a}+\omega) &= \bar{\bar{y}}_1(\bar{a}+\omega), \\ v_5(\bar{a}+\omega-\delta') &= \bar{\bar{y}}_5(\bar{a}+\omega-\delta'), \\ v_6(\bar{a}+\omega+\delta') &= \bar{\bar{y}}_6(\bar{a}+\omega+\delta'). \end{aligned}$$

A root of $\bar{\bar{y}}_1$ lies at $\bar{a}+\omega$. If δ' is taken sufficiently small roots of $\bar{\bar{y}}_5$ and $\bar{\bar{y}}_6$ can be made as close to $\bar{a}+\omega$ as we please. Due to the maximum, however, the situation will be somewhat as here illustrated:

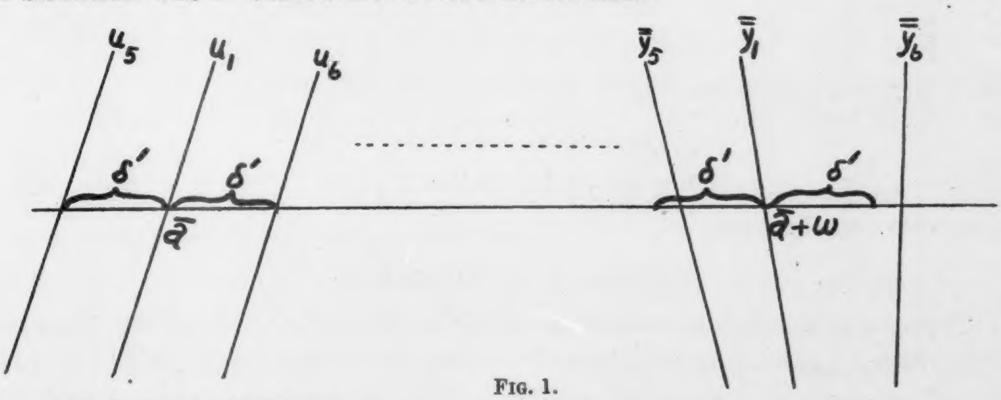

FIG. 1.

We conclude immediately

$$u_5(\bar{a}\pm\delta') v_6(\bar{a}+\omega\pm\delta') - u_6(\bar{a}\pm\delta') \cdot v_6(\bar{a}+\omega\pm\delta') \geqq 0. \qquad (30)$$

One can readily assure himself that another arrangement of the figure will not alter this conclusion. Refer now to (29) and we have that $\Delta(\bar{a}+\delta')\cdot\Delta(\bar{a}-\delta)\geqq 0$, the desired contradiction.

§ 11. SUPPOSE THAT WHEN $\lambda=\Lambda$, $\rho_1=\rho_2$ AND THAT y_7 IS A SOLUTION OF (21), [(21′)] SATISFYING (25) [(25′)] OR (26) [(26′)] SUCH THAT u_7 HAS A ROOT AT $\bar{a}$;* TO PROVE THAT Λ IS NECESSARILY AN EXTREME OF A λ_j, THE TERM EXTREME BEING USED AS PREVIOUSLY.

Under the assumption that $\rho_1=\rho_2$ there exists one solution of (21), y_7 such that $\rho u_7(x)=v_7(x+\omega)$ and a second lineary independent solution y_8 such that $u_8(x+\omega)=C_{21}u_7(x)+\rho u_8(x)$.

$$D(x)=u_7(x)v_8(x+\omega)-u_8(x)v_7(x+\omega)=C_{21}\rho^2[u_7(x)]^2,$$

which obviously never changes sign and does not vanish unless $u_7(x)=0$. We know $u_7(x)$ actually changes sign at $\bar{a}$ and hence when $\delta'>0$ is small enough $D(\bar{a}+\delta')\cdot D(\bar{a}-\delta')>0$.

Assume that Λ is not an extreme of a λ_j. Refer to the determinant (30) discussed in previous section. The corresponding figure now is

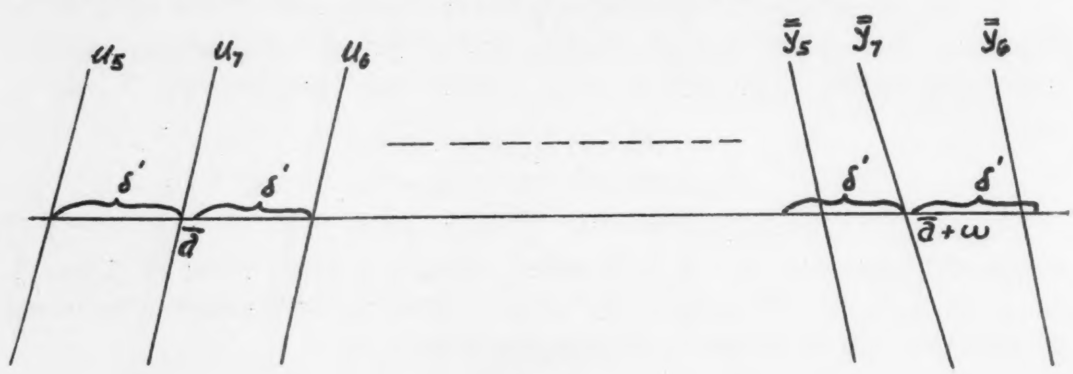

FIG. 2.

from which one concludes that D which is a constant times

$$(u_3(x)v_4(x+\omega)-u_4(x)v_3(x+\omega))$$

does not satisfy the relation $D(\bar{a}+\delta')\cdot D(\bar{a}-\delta')>0$. This is a contradiction and proves the theorem.

§ 12. λ_j A CONSTANT.

Suppose λ_j a constant over some interval $x'\leqq x\leqq x''$. Then by the proof of the previous section λ_j is a value l or a value l'; and, where $\bar{a}$ is any point

* In the case of the difference equation $\bar{a}>a+v$ and in the case of the differential equation $^a>a+\delta$, where δ is as in section 9 (b).

of the interval $x' \leqq x \leqq x''$, a solution satisfying $u_n(\bar{a})=0$ is a solution of the system consisting of (21) [(21')] and (25) [(25')] or of (21) [(21')] and (26) [(26')]. But for definite points $\bar{a}$ sufficiently close together corresponding solutions satisfying $u_n(\bar{a})=0$ are linearly independent. Hence, when $\lambda=\lambda_j$ all solutions of (21) [(21')] satisfy one of these systems. λ_j is then a constant over the whole interval of its definition.

Conversely, if all solutions of (21) [(21')] satisfy (25) [(25')] or (26) [(26')], the corresponding value of λ is necessarily a value λ_j. For, if $\bar{a}$ is any point on the interval $a \leqq x \leqq a+\omega$, those solutions satisfying $u_n(\bar{a})=0$ satisfy $v_n(\bar{a}+\omega)=0$ also.

§ 13. When λ_j is Not a Constant To Prove All the Maxima of λ_j Equal among Themselves and All the Minima Equal among Themselves.

Let $\lambda_j(c)$ not be an extreme. Suppose that $\lambda_j(\bar{x})$ has two minima less than $\lambda_j(c)$. Denote these by $\bar{l}_{j-1}$ and $\bar{\bar{l}}_{j-1}$, $\bar{\bar{l}}_{j-1}<\bar{l}_{j-1}$. Let $\bar{x}$ decrease from c, λ_j can never get less than l_{j-1} for as soon as it equals $\bar{l}_{j-1}$ suppose at $\bar{c}$ then there exists a solution not identically zero of (21), [(21')] satisfying (25) [(25')] or (26) [(26')] which we denote by y_7 where $u_7(c)=0$.

Let y_9 be a solution, not identically zero, satisfying $u_9(c)=0$. Consider $v_7(x+\omega)\,u_9(x)-v_9(x+\omega)\,u_7(x)$. This is a constant times $D(x)$ of section 11. Consequently, it vanishes only when $u_7(x)$ vanishes. But it vanishes at $\bar{c}$ a root of $u_9(x)$. Then by section 11 $\bar{c}$ being a root of $u_7(x)$ is an extreme of λ_j. From the nature of the case it is a minimum; that is, λ_j begins to increase again and the existence of $\bar{\bar{l}}_{j-1}$ is impossible.

§ 14. The Existence of Extremes of the λ_j's.

(a) *The Difference Equation.*

Suppose $\lambda_j(\bar{x})$ $j \geqq 1$ not a constant. Let $\lambda=\lambda_j(a+1)$ and let y_j be a particular solution, not identically zero, satisfying the relation $y_j(a)=0$. y_j has has at least one node on the interval $a<x \leqq a+\omega$. Let a_1 be the first such root to the right of a.

THEOREM: $\lambda_j(a_1+1)=\lambda_j(a+1)$ *and* λ_j *respectively increases or decreases from* a_1 *as* λ_j *increases or decreases from* a.

PROOF: Let $\lambda=\lambda_j(a+1)$. It can be immediately shown as was done in section 10 that $\rho u_j(x)\equiv v_j(x+\omega)$, $\rho \neq 0$ a constant. Consequently, $\lambda_j(a_1+1)=\lambda_j(a+1)$. Next let $0<\bar{x}-a<\delta$, where δ is small, and let $y_{\bar{x}}$ be a solution such that $y_{\bar{x}}(\bar{x})=0$ and $y'_{\bar{x}}(\bar{x})=y'_j(a)$, where the accent denotes the forward derivative. Suppose λ_j increases from a and let $\lambda=\lambda_j(a+1)$. As the nodes

of y_j and $y_{\bar{x}}$ separate each other the first node of $y_{\bar{x}}$ to the right of $\bar{x}$ is to the right of a_1. Let λ increase to $\lambda_j(\bar{x}+1)$ and this node moves still farther to the right. Suppose that when $\lambda=\lambda_j(\bar{x}+1)$ it lies at $\bar{a}_1$ and that $\bar{a}_1-a_1=\eta$. η is arbitrarily small, δ being at our disposal. Now, as in the case of $a+1$ and a_1+1, $\lambda_j(\bar{a}_1+1)=(\lambda_j(\bar{x}+1)$. But $\lambda_j(\bar{x}+1)>\lambda_j(a+1)$. Hence, $\lambda_j(\bar{a}_1+1)>\lambda_j(a_1+1)$ which is what we desired to prove. The following theorem results:

$\lambda_j(\bar{x})$, $j\geqq 1$ *has always at least one minimum and at least one maximum.*

As has been remarked these extremes are the roots of the solutions, y_n, not identically zero, satisfying $u_n(x)\equiv\pm v_n(x+\omega)$. The exact number of these roots on $a+1<x\leqq a+\omega$ will be discussed in the theorem of oscillation, section 17.

(b) *The Differential Equation.*

Only obvious formal changes of the treatment given under (a) are necessary.

$\lambda_j(\bar{x})$, $j\geqq 1$ *has always at least one minimum and at least one maximum.*

§ 15. To Distinguish the Two Cases.

(a) *The Difference Equation.*

Assume $\lambda_j=\lambda_j(\bar{a})$. There exists a solution, not identically zero, which we shall call y_j satisfying the relations $y_j(\bar{a}-1)=0$, $v_j(\bar{a}+\omega)=0$. Let c be that integer such that $c<\bar{a}-1\leqq c+1$ and consider (18) formed for this point.

When λ has any value whatever let y_n denote a solution of (21) satisfying the conditions $y_n(c)=y_j(c)$, $y_n(c+1)=y_j(c+1)$. Then according to section 7 (a) when $a+\omega<x\leqq c+\omega$, $v_n(x+\omega)=y_{11}(x+\omega)$. Let $\eta>0$ be very small.

When $\lambda>0$ is sufficiently large $y_n(c+1+\eta)$ and $y_n(a+\omega+1)$ are of the same sign. Moreover, as shown under section 7, $y_{11}(a+\omega+2)$ and $y_{11}(c+\omega)$ are of the same sign; that is, $v_n(a+\omega+2)$ and $v_n(c+\omega)$ are of the same sign. *To determine, if* $y_n(a+\omega+1)$, *that is,* $u_n(a+\omega+2)$ *is of the same or opposite sign to* $v_n(a+\omega+2)$.

$v_n(a+\omega+2)=y_{10}(a+1)$ [section 7] and as $v_n(a+\omega)=K(a)\Delta y_{10}(a)+by_{10}(a)$ and $\bar{v}_n(a+\omega)=y_{10}(a)$, $K(a)v_n(a+\omega+2)=\bar{D}K(a+\omega)\Delta y_n(a+\omega)+Dy_n(a+\omega)$.

Case α: $\bar{D}=d_{11}+d_{21}(K(a)-b)\neq 0$. Since $\lim\limits_{\lambda=\infty}\dfrac{K(a+\omega)\Delta y_n(a+\omega)}{y_n(a+\omega)}=\infty$ if λ is sufficiently large $v_n(a+\omega+2)$ will have the sign of $\bar{D}K(a+\omega)\Delta y_n(a+\omega)$, that is, of $\bar{D}y_n(c+1+\eta)$.

CASE β: $\overline{D}=0$. $v_n(a+\omega+2)$ has the sign of

$$Dy_n(a+\omega)=[d_{12}+d_{22}(K(a)-b)]y_n(a+\omega),$$

that is of $Dy_n(c+1+\eta)$, for large values of λ.

Now let λ decrease. The number j is the number of times that $v_n(\overline{a}+\omega)$ has already changed sign. *It results that* $u_j(c+2+\eta)=y_j(c+1+\eta)=\pm v_j(c+\omega+2+\eta)$, *and hence*, $\rho=\pm 1$ *as follows:*

CASE α: j even $\overline{D}>0$ $\rho=-1$, $\overline{D}<0$ $\rho=1$;
j odd $\overline{D}>0$ $\rho=1$, $\overline{D}<0$ $\rho=-1$.
CASE β: j even $D>0$ $\rho=-1$, $D<0$ $\rho=1$;
j odd $D>0$ $\rho=1$, $D<0$ $\rho=-1$.

(b) *The Differential Equation.*

Proceed just as under (a). Suppose $\lambda=\lambda_j(\overline{a})$ an extreme of λ_j. Then there exists a solution y_j, not identically zero, satisfying the relations $u_j(\overline{a})=y_j(\overline{a})=0$ and $v_j(\overline{a}+\omega)=0$. Let y_n be a solution when λ has any value satisfying the relations $y_n(\overline{a})=y_j(\overline{a})$ and $y'_n(\overline{a})=y'_j(\overline{a})$. Then by section 7 when $a+\omega\leqq x\leqq\overline{a}+\omega$, $v_n(x+\omega)=y_{11}(x+\omega)$. Moreover, when $\lambda>0$ is sufficiently large and $\overline{\delta}>0$ small, $y_n(\overline{a}+\overline{\delta})$ and $y'_n(a+\omega)$ have the same sign, and $y_{11}(\overline{a}+\omega)$ and $y_{11}(a+\omega+\overline{\delta})$ are of the same sign; that is, $v_n(a+\omega)$ and $v_n(\overline{a}+\omega)$ are of the same sign. To determine then if $u_n(\overline{a}+\overline{\delta})$ and $v_n(\overline{a}+\omega+\delta)$ are of the same sign, is equivalent to determining if $y'_n(a+\omega)$ and $v_n(a+\omega)$ are of the same sign. This can be immediately done as $v_n(a+\omega)=d_{21}K(a+\omega)y'_n(a+\omega)+d_{22}y_n(a+\omega)$ and $\dfrac{K(a+\omega)y'_n(a+\omega)}{y_n(a+\omega)}\to\infty$ as $\lambda\to\infty$.

CASE (α) $d_{21}\neq 0$: $v_n(a+\omega)$ has the same or opposite sign as $y'_n(a+\omega)$ as $d_{21}\gtrless 0$.

CASE (β) $d_{21}=0$: $v_n(a+\omega)$ has the same or opposite sign as $y'_n(a+\omega)$ as $d_{22}\gtrless 0$.

Completing the reasoning as under (a), we have

CASE α: j even $d_{21}>0$ $\rho=-1$, $d_{21}<0$ $\rho=1$;
j odd $d_{21}>0$ $\rho=1$, $d_{21}<0$ $\rho=-1$.
CASE β: j even $d_{22}>0$ $\rho=-1$, $d_{22}<0$ $\rho=1$;
j odd $d_{22}>0$ $\rho=1$, $d_{22}<0$ $\rho=-1$.

§ 16. SOLUTIONS WITHOUT ROOTS.

(a) *The Difference Equation.*

We desire now to study the existence of values l and l', where the corresponding solutions of (21) have no node on $a+1<x<a=\omega$.

Let $\lambda>0$ be very large and let y_1 and y_2 be two solutions determined by the conditions

$$y_1(a)=0, \qquad y_1(a+1)=\frac{1}{K(a)},$$

$$y_2(a)=\frac{1}{K(a)}, \qquad y_2(a+1)=0.$$

Then (see section 2),

$$v_1(a+\omega)=a_{11}u_1(a)+a_{12}u_2(a)=a_{11}\overline{u}_1(a+1)+a_{12}\overline{u}_2(a+1)=(a_{11}-a_{12})\left(1-\frac{b}{K(a)}\right),$$

$$v_1(a+\omega+1)=\overline{v}_1(a+\omega)=a_{11}u_1(a+1)+a_{12}u_2(a+1)=\frac{a_{12}}{K(a)}.$$

Hence,

$$a_{11}=v_1(a+\omega)+(K(a)-b)\overline{v}_1(a+\omega).$$

Similarly,

$$a_{22}=K(a)\overline{v}_2(a+\omega).$$

The characteristic equation is

$$\rho^2-(a_{11}+a_{22})\rho+1=0.$$

$$a_{11}+a_{22}=\overline{D}K(a+\omega)\Delta y_1(a+\omega)+Dy_1(a+\omega)+K(a)d_{21}K(a+\omega)\Delta y_2(a+\omega)$$
$$+d_{22}K(a)y_2(a+\omega).$$

CASE (α): $\overline{D}\neq 0$.

If $\lambda>0$ is sufficiently large, the sign of $a_{11}+a_{22}$ is that of $\overline{D}$. For, write the difference equation in the form

$$K(i+1)\Delta y(i+1)=K(i)\Delta y(i)+G(i)y(i+1),$$

and the facts, that $y_2(a+1)=0$ and $y_1(a+1)=\frac{1}{K(a)}$, and that G becomes infinite with λ shows that by taking λ large enough we can make

$$\frac{K(a+1)\,\Delta y_1(a+1)}{|K(a+1)\Delta y_2(a+1)|}$$

as large we please. An easy induction process shows that the same thing is true of $K(a+\omega)\Delta y_1(a+\omega)$ and $|K(a+\omega)\Delta y_2(a+\omega)|$. Moreover, if y_n is any particular solution whatever, not identically zero, with fixed values at a and $a+1$, $K(a+\omega)\,\Delta y_n(a+\omega)/y_n(a+\omega)$ becomes positively infinite with λ. Hence, $\overline{D}K(a+\omega)\,\Delta y_1(a+\omega)$ will determine the sign of $a_{11}+a_{22}$. In addition when λ is large enough and $i\geqq a$, $K(i)\,\Delta y_n(i)$ increases with i, and, consequently, is greater than zero at $a+\omega$. We conclude as desired that the sign of $a_{11}+a_{22}$ is that of $d_{11}+d_{21}K(a)$ when λ is sufficiently large.

CASE (β). $\overline{D}=0$.

As $K(a+\omega)\Delta y_2(a+\omega)/y_2(a+\omega)$ becomes infinite with λ and as $\Delta y_2(a+\omega)<0$ for sufficiently large values of λ and as from the fact that $d_{11}d_{22}-d_{12}d_{21}=1$, $d_{21}D=-1$, the sign of $a_{11}+a_{22}$ is the sign of D.

Suppose now that for large values of λ $a_{11}+a_{22}>0$; that is, $\overline{D}>0$ or $D>0$ as we have cases (α) or (β). Denote the roots of the characteristic equation by ρ_1 and ρ_2. If λ is sufficiently large, $a_{11}+a_{22}$ is as large as we please, and hence, $\rho_1>0$, say, is large and $\rho_2>0$ small. Let λ decrease $(a_{11}+a_{22})$ is continuous in λ, and hence, ρ_1 and ρ_2 are continuous in λ. Consequently, $\rho_1=\rho_2=1$ before $\rho_1=\rho_2=-1$; that is, there exists a value l larger than the largest value l'. But by section 15 the maximum value of $\lambda_0(\overline{x})$, if such exists, is a value l'. *Hence, if λ_0 has extremes the solutions corresponding to this largest value l can have no node when $a\leqq x\leqq a+\omega$.*

There can not also be a value l' such that a corresponding solution has no node when $a\leqq x\leqq a+\omega$ for by the methods of section 15 solutions without roots always correspond definitely either to a value l or to a value l'.

If for large values of λ $a_{11}+a_{12}<0$ the above situation is exactly reversed and if $\lambda_0(\overline{x})$ has extremes there exists a value l' and no value l such that a corresponding solution has no node, $a\leqq x\leqq a+\omega$.

Consider now the case that $a_{11}+a_{22}>0$ for large positive values of λ. *We wish to prove that there can not be two distinct values l such that corresponding solutions have no node on $a+1\leqq x\leqq a+\omega$.* I find it here necessary to assume K independent of λ an assumption not made in the preceding. Suppose there were two such values l and l'. Denote corresponding solutions by y and $\overline{y}$ and assume that y and $\overline{y}$ are of the same sign when $a+1<x<a+\omega$. We have from (15)

$$K(a+\omega)(y,\overline{y};\ a+\omega,\ a+\omega+1)-K(a)(y,\overline{y};\ a,\ a+1)$$
$$=\sum_{i=a}^{a+\omega-1}(\overline{G}(i)-G(i))\overline{y}(i+1)y(i+1). \tag{31}$$

But as y and $\overline{y}$ both satisfy (25),

$$K(a+\omega)(y,\overline{y};\ a+\omega,\ a+\omega+1)=K(a)(y,\overline{y},\ a,\ a+1). \tag{32}$$

(31) *and* (32) *are inconsistent under the hypothesis that $y(x)$ and $\overline{y}(x)$ preserve the same sign when $a+1<x<a+\omega$.*

If $a_{11}+a_{22}<0$ for large values of λ, we show in exactly the same way that there can not be two values l' such that corresponding solutions have no node.

(b) *The Differential Equation.*

Let y_1 and y_2 be two solutions determined by the conditions

$$y_2(a)=0, \qquad K(a)y_2'(a)=1,$$
$$y_1(a)=\frac{1}{K(a)}, \qquad K(a)y_1'(a)=0.$$

Then

$$v_1(a+\omega)=a_{11}u_1(a)+a_{12}u_2(a)=\frac{a_{11}}{K(a)},$$

$$\bar{v}_2(a+\omega)=a_{21}\bar{u}_1(a)+a_{22}\bar{u}_2(a)=\frac{a_{22}}{K(a)}.$$

Hence,

$$a_{11}+a_{22}=[v_1(a+\omega)+\bar{v}_2(a+\omega)]K(a)=[d_{21}K(a+\omega)y_1'(a+\omega) \\ +d_{22}y_1(a+\omega)+d_{11}K(a+\omega)y_2'(a+\omega)+d_{12}y_2'(a+\omega)]K(a).$$

Now let $\lambda=\bar{\lambda}$ be so large that y_1 and y_2 increase throughout the interval $a<x\leqq a+\omega$,

$$\frac{d}{dx}\frac{y_2}{y_1}=\frac{y_1y_2'-y_2y_1'}{y_1^2}=\frac{1}{K(a)K(x)y_1^2},$$

$$\frac{y_2(\alpha)}{y_1(\alpha)}=\frac{1}{K(a)}\int_a^\alpha\frac{dx}{K(x)y_1^2}\quad a\leqq\alpha\leqq a+\omega.$$

Now let λ increase, when $a<x\leqq a+\omega$ $\frac{1}{y_1^2}$ approaches zero as λ becomes infinite and $1/K(x)$ does not increase. Let $K(x)>\zeta$, $a\leqq x\leqq a+\omega$, when $\lambda\geqq\bar{\lambda}$.

Then $\int_a^\alpha\frac{dx}{K(x)y_1^2}\leqq\frac{1}{\zeta}\int_a^\alpha\frac{dx}{y_1^2}$. But $\frac{1}{(y_1(x))^2}\leqq 1$, and when $a<x\leqq a+\omega$ it approaches zero. It results that

$$\lim_{\lambda=\infty}\frac{y_2(\alpha)}{y_1(\alpha)}=\lim_{\lambda=\infty}\frac{1}{K(a)}\int_a^\alpha\frac{dx}{K(x)(y_1(x))^2}=0.$$

This approach is uniform in α since $\int_a^{a+\omega}\frac{dx}{K(x)(y_1(x))^2}>\int_a^\alpha\frac{dx}{K(x)(y_1(x))^2}$ when $a+\omega<\alpha$.

Now, moreover, from the differential equation

$$\frac{K(a+\omega)y_2'(a+\omega)}{K(a+\omega)y_1'(a+\omega)}=\frac{\int_a^{a+\omega}G(x)y_2(x)\,dx+1}{\int_a^{a+\omega}G(x)y_1(x)\,dx}.$$

Hence, $\frac{y_2'(a+\omega)}{y_1'(a+\omega)}\leqq\int_a^{a+\omega}\frac{y_2(x)}{y_1(x)}dx+\frac{1}{\int_a^{a+\omega}G(x)y_1(x)\,dx}$ when $\lambda\geqq\bar{\lambda}$. Consequently, $\lim_{\lambda=\infty}\frac{y_2'(a+\omega)}{y_1'(a+\omega)}=0$. Hence, since $\lim_{\lambda=\infty}\frac{K(a+\omega)\,y_k'(a+\omega)}{y_k(a+\omega)}=\infty$; $k=1,2$, the sign of $a_{11}+a_{22}$ is that of d_{21} if $d_{21}\neq 0$. If $d_{21}=0$ $d_{22}d_{11}=1$ and the sign of $a_{11}+a_{22}$ is that of d_{22}.

The reasoning is now continued exactly as under (a) with the same result. In order to give the uniqueness proof, using the analogous formula to (31) however, *it is necessary to assume K independent of* λ. There has been no call for this assumption earlier in the paper.

§ 17. THE OSCILLATION THEOREM FOR THE DIFFERENCE EQUATION.*

A solution y_p, where $\lambda_{p-1}(a+2) \geqq l_p \geqq \lambda_p(a+2)$, has $p-1$, p or $p+1$ nodes on the interval $a+1<x<a+\omega$ according to the following table.

$$\text{Let } \frac{K(a+\omega)\,\Delta y_p(a+\omega)}{y_p(a+\omega)} = R_p,$$

1] $y_p(a+\omega) \neq 0$,

$\overline{D} > 0$ and $R_p > -\dfrac{D}{\overline{D}}$,

$p=2m$, p nodes,
$p=2m+1$, $p+1$ nodes;

$\overline{D} < 0$ and $R_p > -\dfrac{D}{\overline{D}}$,

$p=2m$, $p+1$ nodes,
$p=2m+1$, p nodes;

$\overline{D} > 0$ and $R_p < -\dfrac{D}{\overline{D}}$,

$p=2m$, $p-1$ nodes,
$p=2m+1$, p nodes;

$\overline{D} < 0$ and $R_p < -\dfrac{D}{\overline{D}}$,

$p=2m$, p nodes,
$p=2m+1$, $p-1$ nodes;

$\overline{D} = 0$ but $D > 0$,

$p=2m$, p nodes,
$p=2m+1$, $p-1$ nodes;

$\overline{D} = 0$ but $D < 0$,

$p=2m$, $p-1$ nodes,
$p=2m+1$, p nodes.

2] $y_p(a+\omega) = 0$,

$d_{11}+d_{21}K(a) > 0$,

$p=2m$, $p-1$ nodes,
$p=2m+1$, p nodes;

$d_{11}+d_{21}K(a) < 0$,

$p=2m$, p nodes,
$p=2m+1$, $p-1$ nodes.

3] $\overline{D}K(a+\omega)\,\Delta y_p(a+\omega) + D\,y_p(a+\omega) = 0$.

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* The corresponding theorem for the differential equation is not given. It is that given by Birkhoff: *Amer. Math. Soc. Trans.*, Vol. X, p. 269.

This includes all cases not included under 1] and 2]. y_p has p nodes on $a+1<x<a+\omega$ except when $l_p=\lambda_{p-1}(a+2)$ when it has only $p-1$.

PROOF: When λ equals any $\lambda_j(a+2)$, all solutions satisfying $y_n(a+1)=0$ satisfy $\overline{D}K(a+\omega)\,\Delta y_n(a+\omega)+D y_n(a+\omega)=0$ also; for $v_n(a+\omega+2)=\overline{\overline{y}}_n(a+\omega+2)=0$ and $\overline{\overline{y}}_n(a+\omega+2)=\overline{D}K(a+\omega)\,\Delta y_n(a+\omega)+D y_n(a+\omega)$. [See section 7].

Now denote by y_n that solution of (21) satisfying the relations

$$K(a+\omega)\Delta y_n(a+\omega)=D \text{ and } y_n(a+\omega)=-\overline{D}.$$

When $\lambda=l_p$ y has on the interval $a+1<x<a+\omega$, p nodes, or $p-1$ nodes an additional node lying at a. But nodes of two solutions coincide or separate each other, and hence, y_p has on the interval $a+1<x<a+\omega$ either $p-1$, p or $p+1$ nodes.

Now suppose $\overline{D}\neq 0$ and $R_p>-\dfrac{D}{\overline{D}}=R_n$, then y_p has at least as many nodes on the interval $a+1<x<a+\omega$ as y_n has, that is, p or $p+1$. This is a well-known theorem and is assumed. Moreover, from (25)

$$y_p(a+1)=y_p(a+\omega)\overline{D}\left[R_p+\frac{D}{\overline{D}}\right]$$

and consequently $y_p(a+1)$ and $y_p(a+\omega)$ have the same or opposite sign as $\overline{D}\gtrless 0$. The table follows immediately.

If $R_p<-\dfrac{D}{\overline{D}}$, y_p has no more nodes than y_n and the table follows as in the other case. Similarly, the table follows immediately from (25) if $\overline{D}=0$.

If $y_p(a+\omega)=0$ but $\overline{D}\neq 0$ an application of the theorem that the nodes of y_p and y_n separate each other shows that y_p has no more nodes than y_n has on the interval $a+1<x<a+\omega$, that is, not more than p. But from (25) we have the relation $y_p(a+1)=\overline{D}\,y_p(a+\omega+1)$. Hence, the number of nodes of y_p on $a+1<x<a+\omega$ is odd or even as $\overline{D}\gtrless 0$. The table follows.

The statement 3] is immediate and is assumed in what has preceded.

THE UNIVERSITY OF MICHIGAN, *June*, 1915.

Some Singularities of a Contact Transformation.

BY W. V. LOVITT.

§ 1. *Introduction.*

A number of papers have appeared in recent years on the singularities of point transformations. This paper discusses some of the singularities of a contact transformation.

The method of treatment in this paper will follow that of Urner in the *Transactions of the American Mathematical Society*, Vol. XIII, No. 2, pp. 232–264, April, 1912, on Certain Singularities of Point Transformations in Space of Three Dimensions.

In § 2 are given the preliminary hypotheses and notation. In § 3 some preliminary formulae are established. The discussion of the singularities divides itself into three cases, viz.: when the matrix of the Jacobian of the transformation is of rank two, one, or zero. These three cases are treated in §§ 4, 5, and 6, respectively. The irregularities which appear are in the order of contact of the transformed curves. In the latter part of § 5 is given an interesting geometric interpretation of the situation when the matrix of the Jacobian is of rank 1.

§ 2. *Preliminary Hypotheses and Notation.*

Let there be given a real contact transformation

$$x_1=X(x, y, p), \quad y_1=Y(x, y, p), \quad p_1=P(x, y, p). \tag{1}$$

We assume that X, Y, and P are single-valued and continuous when $(x, y, p) = (x_0, y_0, p_0)$ and that they possess with respect to these three variables partial derivatives of the first order also continuous there. The explicit use of any derivative of X, Y, or P shall imply its existence and continuity at the point under consideration, as well as the possession of these properties by all the other derivatives of X, Y, and P of the same and lower orders.

For a proper contact transformation, that is, one which is not a mere point transformation, not both X_p and Y_p can be zero. When we speak of a contact transformation we shall be understood to mean a proper contact transformation.

From the equations (1) we obtain

$$dy_1 - p_1 dx_1 = \rho(x, y, p)(dy - pdx).$$

We shall assume that

$$\rho(x_0, y_0, p_0) = 0,$$

but that

$$\rho(x, y, p) \not\equiv 0.$$

A lineal-element (x_0, y_0, p_0) for which $\rho = 0$ is called a singular or critical lineal-element. It will presently appear that the vanishing of ρ is equivalent to the vanishing of the Jacobian J of the transformation.

We shall study the effect of the above transformation upon the curve, or more precisely the union of lineal-elements, given by

$$x = f(t), \quad y = g(t), \quad y' = g'(t)/f'(t) \equiv h(t), \tag{2}$$

where f and g are continuous, single-valued functions of the real variable t which we assume have continuous derivatives of orders $1, 2, \ldots, k+1$ at the point for which $t = t_0$. With these assumptions h is a single-valued function of the real variable t, continuous together with its derivatives of order $1, 2, \ldots, k$ at the point for which $t = t_0$, except for the zeros of $f'(t)$. We exclude from consideration those values of the parameter t for which $f'(t) = 0$. Then we have certainly one of the quantities $f'(t_0)$, $g'(t_0)$, $h'(t_0)$ different from zero. The curve into which (2) is carried by the transformation is

$$\left.\begin{aligned} x_1 &= X[f(t), g(t), h(t)] \equiv f_1(t), \\ y_1 &= Y[f(t), g(t), h(t)] \equiv g_1(t), \\ p_1 &= P[f(t), g(t), h(t)] \equiv h_1(t). \end{aligned}\right\} \tag{3}$$

Here f_1; g_1, and h_1 are single-valued and continuous and have continuous derivatives of orders $1, 2, \ldots, k$ for $t = t_0$ whenever f, g, and h are so endowed, provided X, Y, and P have continuous partial derivatives of orders $1, 2, \ldots, k$ at the point (x_0, y_0, p_0).

§ 3. *Preliminary Formulae.*

We shall need in the course of this paper to make use of some known relations between X, Y, and P when equations (1) represent a contact transformation. From Lie* we take the formulae

$$[XY] \equiv \begin{vmatrix} X_x, X_y, X_p \\ Y_x, Y_y, Y_p \\ -p, \ 1, \ 0 \end{vmatrix} \equiv 0, \quad [PX] \equiv \begin{vmatrix} X_x, X_y, X_p \\ P_x, P_y, P_p \\ p, -1, \ 0 \end{vmatrix} \equiv \rho, \quad [PY] \equiv \begin{vmatrix} Y_x, Y_y, Y_p \\ P_x, P_y, P_p \\ p, -1, \ 0 \end{vmatrix} \equiv \rho P.$$

$$Y_x = PX_x - p\rho, \quad Y_y = PX_y + \rho, \quad Y_p = PX_p. \tag{4}$$

* "Berührungstransformationen," pp. 68 and 73.

From these relations it follows at once that J and ρ vanish simultaneously. Indeed

$$J \equiv \begin{vmatrix} X_x, & X_y, & X_p \\ Y_x, & Y_y, & Y_p \\ P_x, & P_y, & P_p \end{vmatrix} \equiv \rho[PX] \equiv \rho^2.$$

Let A_1, A_2, A_3; B_1, B_2, B_3; C_1, C_2, C_3 denote, respectively, the cofactors of the elements of the first, second, and third rows in the Jacobian J. Then J may be written in any one of the three following forms:

$$\left.\begin{array}{l} X_xA_1+X_yA_2+X_pA_3, \\ Y_xB_1+Y_yB_2+Y_pB_3, \\ P_xC_1+P_yC_2+P_yC_3. \end{array}\right\} \tag{5}$$

while

$$[XY]\equiv 0, \quad [PX]\equiv\rho, \quad [PY]\equiv\rho P \tag{6}$$

may be written in the respective forms

$$pC_1-C_2\equiv 0, \quad pB_1-B_2\equiv-\rho, \quad pA_1-A_2\equiv\rho P. \tag{7}$$

From (5) and (7) we obtain when $J=0$

$$\begin{cases} (X_x+pX_y)A_1+\ \ X_pA_3=\rho PX_y, \\ (X_x+pX_y)A_2+pX_pA_3=-\rho PX_x, \end{cases}$$

$$\begin{cases} (Y_x+pY_y)B_1+\ \ Y_pB_3=-\rho Y_y, \\ (Y_x+pY_y)B_2+pY_pB_3=\rho Y_x, \end{cases}$$

$$\begin{cases} (P_x+pP_y)C_1+\ \ P_pC_3=0, \\ (P_x+pP_y)C_2+pP_pC_3=0. \end{cases}$$

From equations (4), (6), and (7) and the definitions of A_i, B_i, C_i we find:

$$\begin{aligned}
&B_1\equiv X_pP_y-X_yP_p\equiv\rho_p, \\
&B_2\equiv X_xP_p-X_pP_x=\rho+p\rho_p=\rho+pB_1, \\
&B_3\equiv X_yP_x-X_xP_y=-\rho_x-p\rho_y, \\
&C_1\equiv X_yY_p-X_pY_y=-\rho X_p, \\
&C_2\equiv X_pY_x-X_xY_p=-p\rho X_p=pC_1, \\
&C_3\equiv X_xY_y-X_yY_x=\rho(X_x+pX_y), \\
&A_1\equiv Y_yP_p-Y_pP_y=\rho P_p-P\rho_p=\rho P_p-PB_1, \\
&A_2=p\rho P_p-P(\rho+p\rho_p)=p\rho P_p-PB_2, \\
&A_3=P(\rho_x+p\rho_y)-\rho(P_x+pP_y)=-\rho(P_x+pP_y)-PB_3.
\end{aligned}$$

When $\rho=0$ we obtain

$$C_1=C_2=C_3=0$$

$$\begin{array}{lll} B_1=\rho_p; & B_2=p\rho_p; & B_3=-(\rho_x+p\rho_y), \\ A_1=-PB_1; & A_2=-PB_2; & A_3=-PB_3. \end{array}$$

Whence

$$B_2=pB_1;\quad A_2=-pPB_1; \tag{8}$$
$$X_pB_3=-B_1(X_x+pX_y);\quad X_pA_3=PB_1(X_x+pX_y).$$

§ 4. *Matrix of Rank* 2.

We shall first discuss the singularities of the transformation (1) when the rank of the matrix of the determinant J is 2. From the equations of transformation we compute *

$$y_1'''=\frac{[PX]}{(X_x+X_yy'+X_{y'}y'')^3}y'''+\frac{U(x,y,y',y'')}{(X_x+X_yy'+X_{y'}y'')^3},$$

where the fraction whose numerator is $U(x, y, y', y'')$ stands for all of those terms which do not contain y'''. But

$$[PX]=\rho=0.$$

Then, in general, all curves through the point (x_0, y_0) and with a common value of y' and y'' will be transformed into curves having at the transformed point (x_1^0, y_1^0) a common value of y_1', y_1'', y_1'''. From the above formula we find

$$y_1^{(k)}=\frac{[PX]}{(X_x+X_yy'+X_{y'}y'')^3}y^{(k)}+U_k(x,y,y',\ldots,y^{(k-1)}),$$

where U_k stands for all of those terms which do not contain $y^{(k)}$. From this formula we see that, in general, all curves through the point (x_0, y_0) and with a common value for $y', y'', \ldots, y^{(k-1)}$ will be transformed into curves having at the transformed point (x_1^0, y_1^0) a common value for $y', y'', \ldots, y^{(k)}$. This gives us the

THEOREM: *If a contact transformation*

$$x_1=X(x,y,p),\quad y_1=Y(x,y,p),\quad p_1=P(x,y,p)$$

satisfies the conditions

a) *X, Y, P are of class $C^{(k)}$ at the point P_0 for which $(x, y, p)=(x_0, y_0, p_0)$;*

b) *The functional determinant J vanishes for*

$$x=x_0,\quad y=y_0,\quad z=z_0,$$

but

$$J(x,y,p)\not\equiv 0;$$

c) *The matrix of the determinant J is of rank 2, then curves with contact of order $k-1$ at the point P_0 are transformed into curves with contact of order k at the transformed point $P_0'(x_1^0, y_1^0)$, i. e., the order of contact is increased by one.*

* Lie, "Berührungstransformationen," p. 86.

It may happen, however, that the expression for y_1'',

$$y_1''=\frac{P_x+P_y y'+P_p y''}{X_x+X_y y'+X_p y''}$$

is indeterminate, whence of course $y_1^{(k)}$ becomes indeterminate for $k\geq 2$. In this case, in order to determine the curvature of the transformed curve, we proceed as follows. Let us state here that the slope of the transformed curves is always determined by the equations (1) of transformation. We have

$$dx:dy:dy'=f'(t_0):g'(t_0):h'(t_0),$$

and for the transformed curve

$$dx_1:dy_1:dy_1'=f_1'(t_0):g_1'(t_0):h_1'(t_0)$$

where

$$\left.\begin{aligned} f_1'&=X_xf'+X_yg'+X_ph',\\ g_1'&=Y_xf'+Y_yg'+Y_ph',\\ h_1'&=P_xf'+P_yg'+P_ph'.\end{aligned}\right\} \tag{9}$$

From these formulae we can determine the slope and radius of curvature of the transformed curve unless

$$f_1'(t_0)=g_1'(t_0)=h_1'(t_0)=0.$$

Since the matrix of the determinant of J is of rank 2, there exists one direction and radius of curvature for the original curve and only one for which the determination (9) fails. This is defined by the ratios

$$f':g':h'=I_1:I_2:I_3,$$

where I stands for that one of the letters A, B, C for which $I_1:I_2:I_3\neq 0:0:0$. We shall call this the *critical curvature* for the lineal-element (x_0, y_0, p_0).

Let us now assume that the curve (2) has the critical curvature. Suppose, however, that not all of the quantities $f_1''(t_0)$, $g_1''(t_0)$, $h_1''(t_0)$ are zero. Then

$$f_1'(t)\equiv f_1''[t_0+\theta_1(t-t_0)](t-t_0), \qquad (0<\theta_1<1)$$

with similar formulae for g_1', h_1'. Hence allowing t to approach t_0, we see that

$$dx_1:dy_1:dp_1=f_1''(t_0):g_1''(t_0):h_1''(t_0).$$

Placing $f'=kI_1$, $g'=kI_2$, $h'=kI_3$ we have

$$\left.\begin{aligned} f_1''&=X_xf''+X_yg''+X_ph''+k^2U^2(X, I),\\ g_1''&=Y_xf''+Y_yg''+Y_ph''+k^2U^2(Y, I),\\ h_1''&=P_xf''+P_yg''+P_ph''+k^2U^2(P, I),\end{aligned}\right\} \tag{10}$$

where

$$U^2(FI)\equiv I_1^2F_{xx}+I_2^2F_{yy}+I_3^2F_{pp}+2I_2I_3F_{yp}+2I_3I_1F_{px}+2I_1I_2F_{xy}.$$

If we make a change of parameter, $t'=kt$, the functions f, g, h go over into

such functions $\bar{f}, \bar{g}, \bar{h}$ of t' that when $t'=kt_0$, $\bar{f}=I_1$, $\bar{g}=I_2$, $\bar{h}=I_3$. Then, let us drop dash and prime, merely supposing the necessary changes already made in the functions f, g, h as they stand. This done we have a right to set k in equations (10) equal to unity and $U^2(X, I)$, $U^2(Y, I)$, $U^2(P, I)$ become constants which we shall designate, respectively, by ρ_2, σ_2, τ_2. The curvature of the transformed curve is now determined by $f_1'' : h_1''$ unless

$$f_1'' : h_1'' = 0 : 0.$$

In this case the equations

$$\left.\begin{aligned} X_x f'' + X_y g'' + X_p h'' + \rho_2 = 0, \\ P_x f'' + P_y g'' + P_p h'' + \tau_2 = 0, \end{aligned}\right\} \tag{11}$$

are independent as equations for the determination of f'', g'', h''. At the point under consideration

$$I_1 = f', \; I_2 = g', \; I_3 = h' = \frac{f'g'' - g'f''}{f'^2} \tag{12}$$

We may now choose $(I_1, I_2, I_3 = B_1, B_2, B_3)$. By hypothesis $f' \neq 0$. Then $B_1 \neq 0$ and, hence, it follows from equations (8) that $B_2 \neq 0$. We are thus furnished with a third equation in f'', g'', h'':

$$B_2 f'' - B_1 g'' = -B_3 B_1^2. \tag{13}$$

The three equations (11) and (13) are independent and from them can be found the values of f'', g'', h'' for which the determination of the curvature of the transformed curve from equations (10) fails.

From the last of equations (12) by successive differentiations we obtain

$$h^{(j)} = \frac{f'g^{(j)} - g'f^{(j)}}{f'^2} + H_j, \tag{14}$$

where H_j depends upon the derivatives of f, g, of orders $1, 2, \ldots, j-1$. When $f' = B_1$ and $g' = B_2$ the equation (14) may be written in the form

$$B_2 f^{(j)} - B_1 g^{(j)} = B_1^2 (H_j - h^{(j-1)}). \tag{15}$$

The General Case. Let us now suppose that there exists a curve with the lineal-element (x_0, y_0, p_0) for which $f^{(j)}, h^{(j)}$ have such values that

$$f_1^{(j)}(t_0) = h_1^{(j)}(t_0) = 0,$$

for $j = 1, 2, \ldots, k-1$, but $[f_1^{(k)}]^2 + [h_1^{(k)}]^2 \neq 0$. Then let us join the point $t = t_0$ on the transformed curve with the point (x_1, y_1) for which $t = t_0 + \varepsilon$, by means of a secant. Then

$$x_1 - x_1^0 = f_1(t_0 + \varepsilon) - f_1(t_0) = \frac{\varepsilon^k}{k!} f_1^{(k)}[t_0 + \theta_1 \varepsilon], \qquad (0 < \theta_1 < 1),$$

with similar formulae for $y_1 - y_1^0$ and $p_1 - p_1^0$. Now, if we allow ε to approach

zero, we obtain for the determination of the curvature of the transformed curve

$$dx_1 : dy_1' = dx_1 : dp_1 = f_1^{(k)}(t_0) : h_1^{(k)}(t_0).$$

If there exists a curve satisfying these requirements, then we shall say that (x_0, y_0, p_0) is a *singular lineal-element* of order $k-1$ at least. The order will be exactly $k-1$ if there is no curve for which

$$f_1^{(j)} = h_1^{(j)} = 0 \quad \text{for} \quad j = 1, 2, \ldots, k.$$

That is, the order of the singularity is precisely one unit less than the order of the highest derivatives which must be used to find the curvature of the transform of every curve with the lineal-element (x_0, y_0, p_0). Assuming for the moment that we have to do with a singular lineal-element of order $k-1$, let us designate as *critical curves* that family of curves which yield the values of $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ necessary to make

$$f_1^{(j)} = h_1^{(j)} = 0 \quad \text{for} \quad j = 1, 2, \ldots, k-1.$$

By differentiating equations (9) $j-1$ times we get

$$\left.\begin{aligned} f_1^{(j)}(t) &= X_x f^{(j)}(t) + X_y g^{(j)}(t) + X_p h^{(j)}(t) + \rho_j, \\ g_1^{(j)}(t) &= \qquad\qquad\qquad " \qquad\qquad\qquad + \sigma_j, \\ h_1^{(j)}(t) &= \qquad\qquad\qquad " \qquad\qquad\qquad + \tau_j. \end{aligned}\right\} \tag{16}$$

The functions ρ_j, σ_j, τ_j depend on the derivatives of X, Y, P of orders $2, \ldots, j$ and on the derivatives of f, g, h of orders $1, 2, \ldots, j-1$. Then the critical curves have the curvature of their corresponding transformed curves determined by $f_1^{(k)}(t_0) : h_1^{(k)}(t_0)$. In computing these we must evaluate ρ_k, τ_k and these depend upon the derivatives of f, g, h of orders $1, 2, \ldots, k-1$, taken at $t = t_0$. The values of these derivatives are determined uniquely by the equations

$$\left.\begin{aligned} &f_1^{(j)} = \ h_1^{(j)} = 0, \\ &B_2 f^{(j)} - B_1 g^{(j)} = B_1^2 (H_j - h^{(j-1)}), \qquad (j = 1, 2, \ldots, k-1). \end{aligned}\right\} \tag{15}$$

It is clear that all of the critical curves have with each other contact of order k at least. It is also clear from equations (16), if we put $j = k$, that the critical curves which have contact of order $k+1$ are transformed into curves with second order contact. Let us now assign arbitrary values to $f_1^{(k)}$, $g_1^{(k)}$, $h_1^{(k)}$, then the first and last of equations (16) and equations (15), all written for $j = k$, are three independent equations for the determination of $f^{(k)}$, $g^{(k)}$, $h^{(k)}$. We may now summarize our results as follows:

THEOREM: *All of those curves for which* $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ *vanish, but for which* $[f^{(j+1)}]^2 + [g^{(j+1)}]^2 + [h^{(j+1)}]^2 \neq 0$ *are transformed into curves with a common*

direction and curvature. The curvatures of the transformed curves for different values of j are in general different.

§ 5. *Matrix of Rank* 1.

We will now consider the case in which the matrix of the Jacobian:

$$J \equiv \begin{vmatrix} X_x, & X_y, & X_p \\ Y_x, & Y_y, & Y_p \\ P_x, & P_y, & P_p \end{vmatrix}$$

of the transformation (1) is of rank 1. Let X_i, Y_i, P_i be a column of elements, not all of which vanish at (x_0, y_0, p_0), while i_x, i_y, i_p is a row possessing this property. The union of lineal-elements (2) is transformed by means of equations (1) into the union (3) and tangent curves go into tangent curves. If we desire to find the curvature of the transformed curve we find that it becomes indeterminate in form if, and only if, both f'_1 and h'_1 vanish simultaneously. Our problem is to evaluate this indeterminate form under the different circumstances which may arise.

In general, the direction and curvature of the transformed curve can be found from

$$dx_1 : dy_1 : dy'_1 = f'_1(t_0) : g'_1(t_0) : h'_1(t_0),$$

where

$$\left.\begin{aligned} f'_1 &= X_x f' + X_y g' + X_p h', \\ g'_1 &= Y_x f' + Y_y g' + Y_p h', \\ h'_1 &= P_x f' + P_y g' + P_p h'. \end{aligned}\right\} \tag{9}$$

The determination of these ratios will fail if, and only if, such a curve be taken that

$$X_x f' + X_y g' + X_p h' = 0. \tag{17}$$

All other curves then will be transformed into curves for which

$$dx_1 : dy_1 : dy'_1 = X_i : Y_i : P_i,$$

and we may choose $X_i, Y_i, P_i = X_p, Y_p, P_p$. For $X_p \neq 0$, otherwise from the equation $Y_p - PX_p = 0$ we would have $Y_p = 0$ and the transformation would reduce to a mere point transformation which has been excluded. The equation (17) may be written in the form

$$X_p y'' + X_y y' + X_x = 0. \tag{18}$$

Since $X_p \neq 0$, this equation determines y'' and thus a definite radius of curvature is determined for certain plane curves with the common lineal-element (x_0, y_0, p_0). *If we avoid curves with the radius of curvature thus determined, and having the common lineal-element* (x_0, y_0, p_0), *then equations* (9) *determine for us the direction and curvature of the transformed curves.*

We are considering only those values of p for which $J=0$ and the matrix of J is of rank 1. Under these conditions we find $\rho_x+p\rho_y=0$. This yields a second homogeneous relation between f', g', h':

$$f'\rho_x+g'\rho_y=0. \tag{19}$$

From equations (17) and (19) we find

$$\begin{aligned} f' &= -\lambda X_p\rho_y \equiv \lambda\alpha, \\ g' &= \lambda X_p\rho_x \equiv \lambda\beta, \\ h' &= \lambda(X_x\rho_y - X_y\rho_x) \equiv \lambda\gamma, \end{aligned}$$

where α, β, γ are defined by these equations and λ is a factor of proportionality. The curvature of the transformed curve is then determined unless

$$f':g':h'=\alpha:\beta:\gamma.$$

Any curve for which $f':g':h'=\alpha:\beta:\gamma$ will be spoken of as a curve with the *critical curvature.* If a curve has this critical curvature, then, in general, the direction and curvature of the transformed curve can be found from f_1'', g_1'', h_1''. Placing $f'=\lambda\alpha$, $g'=\lambda\beta$, $h'=\lambda\gamma$, we have

$$\left.\begin{aligned} f_1'' &= X_xf''+X_yg''+X_ph''+\lambda^2U^2(X, \alpha, \beta, \gamma), \\ g_1'' &= Y_xf''+Y_yg''+Y_ph''+\lambda^2U^2(Y, \alpha, \beta, \gamma), \\ h_1'' &= P_xf''+P_yg''+P_ph''+\lambda^2U^2(Y, \alpha, \beta, \gamma), \end{aligned}\right\} \tag{20}$$

where

$$U^2(F, \alpha, \beta, \gamma)=\alpha^2F_{xx}+\beta^2F_{yy}+\gamma^2F_{pp}+2\beta\gamma F_{yp}+2\gamma\alpha F_{px}+2\alpha\beta F_{xy}.$$

If we make a change of parameter $t'=\lambda t$, the functions f, g, h go over into such functions $f, \bar{g}, \bar{h}$ of t' that when $t'=\lambda t_0$, $\bar{f}'=\alpha$, $\bar{g}'=\beta$, $\bar{h}'=\gamma$. Then let us drop dash and prime merely supposing the necessary changes already made in the functions f, g, h as they stand. This done we have a right to set λ in equations (20) equal to unity. Let us now put

$$\begin{aligned} \rho_2 &= U^2(X, \alpha, \beta, \gamma), \\ \sigma_2 &= U^2(Y, \alpha, \beta, \gamma), \\ \tau_2 &= U^2(P, \alpha, \beta, \gamma). \end{aligned}$$

We now consider ρ_2, σ_2, τ_2 as completely determined. The variable portions of the right members of (20) as f'', g'', h'' take on all possible values, preserve throughout, the ratios $X_i:Y_i:P_i$ so that

$$\begin{aligned} f_1'' &= \lambda X_i+\rho_2, \\ g_1'' &= \lambda Y_i+\sigma_2, \\ h_1'' &= \lambda P_i+\tau_2. \end{aligned}$$

Elimination of λ will give the following three equations:

$$Y_i(f_1''-\rho_2)-X_i(g_1''-\sigma_2)=0,$$
$$P_i(g_1''-\sigma_2)-Y_i(h_1''-\tau_2)=0,$$
$$X_i(h_1''-\tau_2)-P_i(f_1''-\rho_2)=0,$$

only two of which are independent. Between these two, the terms independent of f_1'', g_1'', h_1'' may be eliminated to yield a homogeneous linear relation. In particular, if $Y_i\neq 0$, this equation will be

$$(P_i\sigma_2-Y_i\tau_2)(Y_if_1''-X_ig_1'')-(Y_i\rho_2-X_i\sigma_2)(P_ig_1''-Y_ih_1'')=0. \tag{21}$$

It will be noticed that (21) is satisfied by putting

$$f_1'':g_1'':h_1''=X_i:Y_i:P_i, \tag{22}$$

whatever be the values of ρ_2, σ_2, τ_2.

The curvature of the transformed curve is now determined unless $f_1''=g_1''=h_1''=0$. We state this result as follows:

THEOREM: *All curves for which f', g', h', f'', g'', h'' have values which render $f_1'=g_1'=h_1'=0$, but for which not all f_1'', g_1'', h_1'' are zero are transformed into tangent curves at the transformed point (x_1^0, y_1^0) and all the transformed curves have the same radius of curvature at the transformed point.*

The vanishing of the three quantities

$$Y_i\rho_2-X_i\sigma_2,\quad P_i\sigma_2-Y_i\tau_2,\quad X_i\tau_2-P_i\rho_2$$

is the condition that the three equations

$$\left.\begin{aligned} X_xf''+X_yg''+X_ph''+\rho_2=0,\\ Y_xf''+Y_yg''+Y_ph''+\sigma_2=0,\\ P_xf''+P_yg''+P_ph''+\tau_2=0,\end{aligned}\right\} \tag{23}$$

be consistent. If, then, we have

$$Y_i\rho_2-X_i\sigma_2=P_i\sigma_2-Y_i\tau_2=X_i\tau_2-P_i\rho_2=0,$$

there exist curves having the critical curvature, such that $f_1''=g_1''=h_1''=0$, and, therefore, our determination of curvature for the transformed curve is no longer valid.

Let us suppose that f'', g'', h'' have such values that equations (23) are satisfied. No two of these equations are independent as the matrix of J is now one. The curvature of the transformed curve will be given by $f_1''':g_1''':h_1'''$, where

$$\left.\begin{aligned} f_1'''=X_xf'''+X_yg'''+X_ph'''+\rho_3,\\ g_1'''=Y_xf'''+Y_yg'''+Y_ph'''+\sigma_3,\\ h_1'''=P_xf'''+P_yg'''+P_ph'''+\tau_3,\end{aligned}\right\} \tag{24}$$

where

$$\begin{aligned}\rho_3 \equiv f''\,[3X_{xx}f'+3X_{xy}g'+3X_{xp}h'] \\ +g''\,[Y_{yx}f'+Y_{yy}g'+Y_{yp}h'+2X_{yy}g'+2X_{yp}h'+2X_{xy}f'] \\ +h''\,[P_{xp}f'+P_{yp}g'+P_{pp}h'+2X_{pp}h'+2X_{yp}g'+2X_{xp}f']+\bar{\rho}_3,\end{aligned}$$

where $\bar{\rho}_3$ is a function of f', g', h', and third derivatives of X, Y, P, with similar expressions for σ_3, τ_3, viz., expressions which are *linear* in f'', g'', h''.

Equations (24) as f''', g''', h''', take on all possible values may be written in the form

$$\begin{aligned}f_1''' &= \lambda_3 X_i+\rho_3, \\ g_1''' &= \lambda_3 Y_i+\sigma_3, \\ h_1''' &= \lambda_3 P_i+\tau_3, \ \lambda_3 \text{ being variable.}\end{aligned}$$

Elimination of λ_3 will give the following three equations:

$$\begin{aligned}Y_i(f_1'''-\rho_3)-X_i(g_1'''-\sigma_3)=0, \\ P_i(g_1'''-\sigma_3)-Y_i(h_1'''-\tau_3)=0, \\ X_i(h_1'''-\tau_3)-P_i(f_1'''-\rho_3)=0,\end{aligned}$$

only two of which are independent. Between these two, the terms independent of f_1''', g_1''', h_1''', may be eliminated, to yield a homogeneous linear relation. In particular, if $Y_i \neq 0$, this equation will be

$$(P_i\sigma_3-Y_i\tau_3)(Y_if_1'''-X_ig_1''')-(Y_i\rho_3-X_i\sigma_3)(P_ig_1'''-Y_ih_1''')=0.$$

It will be noticed that this equation is satisfied by putting

$$f_1''' : g_1''' : h_1''' = X_i : Y_i : P_i \tag{25}$$

whatever be the values of ρ_3, σ_3, τ_3. If

$$Y_i\rho_3-X_i\sigma_3=P_i\sigma_3-Y_i\tau_3=X_i\tau_3-P_i\rho_3=0, \tag{26}$$

there exist curves having the given critical curvature such that $f_1'''=g_1'''=h_1'''=0$, and, therefore, our determination of curvature of the transformed curve from (24) is no longer valid. Equations (26) are *linear* equations in f'', g'', h'', and one of them is a consequence of the other two. There are two additional relations between f'', g'', h'':

$$X_xf''+X_yg''+X_ph''+\rho_2=0$$

and

$$B_2f''-B_1g''=-B_3B_1^2.$$

We have, thus, four independent equations for the determination of f'', g'', h''. In general, they cannot be satisfied. In case that these equations are satisfied f'', g'', h'' are uniquely determined and ρ_3, σ_3, τ_3 become certain fixed constants. The curvature of the transformed curve is now determined from $f_1^{(4)}$, $g_1^{(4)}$, $h_1^{(4)}$.

General Case. We now proceed to the discussion of the general case. Let us suppose that there exists a curve with the lineal-element (x_0, y_0, p_0) for which $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ have such values that

$$f_1^{(j)}(t_0) = g_1^{(j)}(t_0) = h_1^{(j)}(t_0) = 0, \qquad (j=1, \ldots, k-1),$$

but

$$[f_1^{(k)}]^2 + [g_1^{(k)}]^2 + [h_1^{(k)}]^2 \neq 0.$$

Then the direction of the transformed curve will be given by

$$\left.\begin{aligned} f_1^{(k)} &= X_x f^{(k)}(t) + X_y g^{(k)}(t) + X_p h^{(k)}(t) + \rho_k, \\ g_1^{(k)} &= Y_x f^{(k)}(t) + Y_y g^{(k)}(t) + Y_p h^{(k)}(t) + \sigma_k, \\ h_1^{(k)} &= P_x f^{(k)}(t) + P_y g^{(k)}(t) + P_p h^{(k)}(t) + \tau_k. \end{aligned}\right\} \tag{27}$$

The functions ρ_k, σ_k, τ_k, depend upon the derivatives of X, Y, P, of orders $2, \ldots, k$, and on the derivatives of f, g, h, of orders $1, 2, \ldots, k-1$, and are *linear* in $f^{(k-1)}$, $g^{(k-1)}$, $h^{(k-1)}$.

In computing $f_1^{(k)}(t_0)$, $g_1^{(k)}(t_0)$, $h_1^{(k)}(t_0)$, we must evaluate ρ_k, σ_k, τ_k, and these depend upon the derivatives of f, g, h, of orders $1, 2, \ldots, k-1$ taken at $t=t_0$. The values of these derivatives are uniquely determined by the equations

$$\begin{gathered} Y_i\rho_j - X_i\sigma_j = P_i\sigma_j - Y_i\tau_j = X_i\tau_j - P_i\rho_j = 0, \\ f_1^{(j)} = g_1^{(j)} = h_1^{(j)} = 0, \qquad (j=1, 2, \ldots, k-1). \end{gathered}$$

Equations (27), as $f^{(k)}$, $g^{(k)}$, $h^{(k)}$ take on all possible values, may be written in the form

$$\begin{aligned} f_1^{(k)} &= \lambda_k X_i + \rho_k, \\ g_1^{(k)} &= \lambda_k Y_i + \sigma_k, \\ h_1^{(k)} &= \lambda_k P_i + \tau_k, \quad \lambda_k \text{ being variable.} \end{aligned}$$

Elimination of λ_k will give the following three equations:

$$\begin{aligned} Y_i(f_1^{(k)} - \rho_k) - X_i(g_1^{(k)} - \sigma_k) &= 0, \\ P_i(g_1^{(k)} - \sigma_k) - Y_i(h_1^{(k)} - \tau_k) &= 0, \\ X_i(h_1^{(k)} - \tau_k) - P_i(f_1^{(k)} - \rho_k) &= 0, \end{aligned}$$

only two of which are independent. Between these two, the terms independent of $f_1^{(k)}$, $g_1^{(k)}$, $h_1^{(k)}$, may be eliminated, to yield a homogeneous linear relation. In particular, if $Y_i \neq 0$, this equation will be

$$(P_i\sigma_k - Y_i\tau_k)(Y_i f_1^{(k)} - X_i g_1^{(k)}) - (Y_i\rho_k - X_i\sigma_k)(P_i g_1^{(k)} - Y_i h_1^{(k)}) = 0.$$

It will be noticed that this equation is satisfied by putting

$$f_1^{(k)} : g_1^{(k)} : h_1^{(k)} = X_i : Y_i : P_i \tag{28}$$

whatever be the values of ρ_k, σ_k, τ_k.

An inspection of equations (22), (25) and (28) shows that the following theorem is true:

THEOREM: *Let there be given a lineal-element* (x_0, y_0, p_0) *for which the Jacobian of the transformation is of rank* 1. *Then, all curves, possessing the common lineal-element* (x_0, y_0, p_0) *will be transformed by means of equations* (1) *into curves, which at the transformed point* (x_1^0, y_1^0) *have a common tangent, and in addition a common radius of curvature given by*

$$dx_1 : dy_1 : dy_1' = X_i : Y_i : P_i .$$

In case the matrix of the Jacobian J, of the transformation is of rank 1, we find from the preliminary formulae

$$\rho(x, y, p) = 0, \quad \rho_p = 0, \quad \rho_x + p\rho_y = 0. \tag{29}$$

Now, $\rho(x, y, p) = 0$ is a differential equation.

Equations (29) *assure us that the critical lineal-element is one whose direction and point coincide with that of the tangent and point of tangency to the curve of the singular solution of the differential equation* $\rho = 0$.

Let us now examine more closely the equation

$$X_p y'' + X_y y' + X_x = 0. \tag{18}$$

We are talking about a particular lineal-element and hence y' is fixed. This equation determines y'', since $X_p \neq 0$. But if y' and y'' are fixed, then the radius of curvature is fixed. *We have then associated with each lineal-element of the singular solution curve a definite radius of curvature.* Denote by (α, β) the coordinates of the center of curvature and by (x, y) a point on the singular solution curve. The locus of the centers of critical curvature will be given by eliminating x, y, y', y'' from the following equations:

$$\rho(x, y, p) = 0, \quad \rho_p(x, y, p) = 0,$$
$$X_p y'' + X_y y' + X_x = 0,$$
$$\alpha = x - \frac{y'(1 + y'^2)}{y''}, \quad \beta = y + \frac{1 + y'^2}{y''}.$$

Illustration. $X = p,$

$$Y = \tfrac{1}{2}(y - px)^2 - p^2(y - px),$$
$$P = -2p(y - px) - x(y - px - p^2).$$

$$J = \begin{vmatrix} 0 & 0 & 1 \\ -p(y - px - p^2), & y - px - p^2, & -x(y - px - p^2) - 2p(y - px) \\ 2p^2 + px - (y - px - p^2), & -2p - x, & x^2 + 6px - 2y \end{vmatrix} = (y - px - p^2)^2 = \rho^2$$

whence

$$\rho = y - px - p^2.$$

The singular solution of this equation is

$$x^2+4y=0.$$

The matrix of the Jacobian J is of rank 1 at any lineal-element given by

$$(x, -\tfrac{1}{4}x^2, -\tfrac{1}{2}x), \tag{30}$$

or, in particular, at $(2, -1, -1)$. For any lineal-element given by (30) we have

$$\rho=\rho_p=\rho_x+p\rho_y=0.$$

The determination of the radius of curvature of the transformed curves from equations (9) fails if, and only if,

$$X_p y''+X_y y'+X_x=0. \tag{18}$$

In the present instance this equation reduces to

$$y''=0.$$

This makes the radius of curvature of the original curves infinite. Then, the critical curves (C) which pass through any point P of the envelope curve (E) are those curves which have the tangent to (E) at P for an inflectional tangent.

§ 6. *Matrix of Rank Zero.*

This case is impossible for a proper contact transformation. For, if $X_p=0$, it follows from the preliminary formulae that $Y_p=0$ also, and the transformation reduces to a mere point transformation.

Oscillations near an Isosceles-Triangle Solution of the Problem of Three Bodies as the Finite Masses Become Unequal.

By Daniel Buchanan.

§ 1. *Introduction.*

In an article entitled "Oscillations near one of the Isosceles-Triangle Solutions of the Three-Body Problem," which appeared in the July (1915) number of the *Proceedings of the London Mathematical Society*, the author of the present paper discussed the periodic oscillations of an infinitesimal body about a straight line drawn through the centre of gravity of two finite bodies of equal mass and perpendicular to the plane of their motion, which was assumed to be circular. In the problem now under consideration the finite bodies are assumed to be of unequal mass and the third body is assumed to be infinitesimal. The finite bodies are started so that they move in circles, and the infinitesimal body oscillates about the straight line through the centre of mass of the finite bodies and perpendicular to the plane of their motion, as in the former article. Initial conditions are determined so that the oscillations shall be periodic. The solutions for the motion of the infinitesimal body are expansible as power series in a certain parameter ϵ which represents half the difference in mass between the finite bodies. When $\epsilon=0$ the solutions reduce to the simplest case of the isosceles-triangle solutions, viz., that in which the finite bodies are of equal mass and move in circles and the third body is infinitesimal.

As we shall have frequent occasion to refer to the former paper we shall refer to it as *Proc.*, followed by the number of the equations or the section.

§ 2. *The Differential Equations.*

Let m_1 and m_2 denote the two finite bodies of mass $m-\epsilon$ and $m+\epsilon$, respectively, and let μ denote the infinitesimal body. Let the unit of mass be so chosen that $m=1/2$, the linear unit so that the distance from m_1 to m_2 shall be unity, and the unit of time so that the Gaussian constant shall also be unity. Let the

origin of coordinates be taken at the centre of mass, the plane of motion of m_1 and m_2 as the $\xi\eta$-plane, and let the coordinates of m_1, m_2 and μ be denoted by ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2 and ξ, η, ζ, respectively. If the masses m_1 and m_2 are started from the points $1/2+\varepsilon$, 0, 0, and $-1/2+\varepsilon$, 0, 0, respectively, so that they will move in circles, then

$$\left.\begin{aligned}\xi_1&=(\tfrac{1}{2}+\varepsilon)\cos(t-t_0), & \xi_2&=-(\tfrac{1}{2}-\varepsilon)\cos(t-t_0),\\ \eta_1&=(\tfrac{1}{2}+\varepsilon)\sin(t-t_0), & \eta_2&=-(\tfrac{1}{2}-\varepsilon)\sin(t-t_0),\end{aligned}\right\}\qquad(1)$$

and the differential equations of motion for the infinitesimal body are

$$\left.\begin{aligned}\xi''&=-\frac{\xi}{2}\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+\frac{1}{2}\left[\frac{\xi_1}{\rho_1^3}+\frac{\xi_2}{\rho_2^3}\right]+\varepsilon\xi\left[\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right]-\varepsilon\left[\frac{\xi_1}{\rho_1^3}-\frac{\xi_2}{\rho_2^3}\right],\\ \eta''&=-\frac{\eta}{2}\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+\frac{1}{2}\left[\frac{\eta_1}{\rho_1^3}+\frac{\eta_2}{\rho_2^3}\right]+\varepsilon\eta\left[\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right]-\varepsilon\left[\frac{\eta_1}{\rho_1^3}-\frac{\eta_2}{\rho_2^3}\right],\\ \zeta''&=-\frac{\zeta}{2}\left[\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right]+\varepsilon\zeta\left[\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right],\\ \rho_1&=[(\xi-\xi_1)^2+(\eta-\eta_1)^2+\zeta^2]^{\frac{1}{2}},\\ \rho_2&=[(\xi-\xi_2)^2+(\eta-\eta_2)^2+\zeta^2]^{\frac{1}{2}}.\end{aligned}\right\}\qquad(2)$$

If we refer the motion of the system to a set of rectangular coordinates rotating about the ζ-axis with the uniform velocity unity, that is, if we transform the coordinates of μ by the substitutions

$$\left.\begin{aligned}\xi&=x\cos(t-t_0)-y\sin(t-t_0),\\ \eta&=x\sin(t-t_0)+y\cos(t-t_0),\\ \zeta&=z,\end{aligned}\right\}\qquad(3)$$

then, after (1) and (3) are substituted in (2), the differential equations become

$$\left.\begin{aligned}x''-2y'-x&=-\frac{x}{2}\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]+\left[\frac{1}{4}+\varepsilon x-\varepsilon^2\right]\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right],\\ y''+2x'-y&=-\frac{y}{2}\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]+\varepsilon y\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right],\\ z''&=-\frac{z}{2}\left[\frac{1}{r_1^3}+\frac{1}{r_2^3}\right]+\varepsilon z\left[\frac{1}{r_1^3}-\frac{1}{r_2^3}\right],\\ r_1&=[(\tfrac{1}{2}+\varepsilon-x)^2+y^2+z^2]^{\frac{1}{2}},\quad r_2=[(\tfrac{1}{2}-\varepsilon+x)^2+y^2+z^2]^{\frac{1}{2}}.\end{aligned}\right\}\qquad(4)$$

These equations admit the Jacobian integral

$$(x')^2+(y')^2+(z')^2=x^2+y^2+\frac{1-2\varepsilon}{r_1}+\frac{1+2\varepsilon}{r_2}+\text{const.}\qquad(5)$$

When $\varepsilon=0$ and the infinitesimal is projected along the z-axis, then $x\equiv y\equiv 0$ and the differential equations (4) reduce to

$$z''=-\frac{8z}{(1+4z^2)^{\frac{3}{2}}}.$$

The periodic solution of this equation is, *Proc.* (4),

$$z=\psi=a\sin(\tau-\tau_0)+\frac{3}{16}a^3[\sin 3(\tau-\tau_0)-\sin(\tau-\tau_0)]+\ldots,$$

$$\tau-\tau_0=\frac{(t-t_0)}{\sqrt{\frac{1}{8}(1+\delta)}},\quad \delta=\frac{9}{2}a^2-\frac{141}{32}a^4+\frac{35}{2}a^6+\ldots.$$

The constant a is a variable parameter and $a/\sqrt{\{\frac{1}{8}(1+\delta)\}}$ denotes the initial projection of μ from the origin when $\varepsilon=0$. The numerical values of t_0 and τ_0 can both be taken to be zero without loss of generality. As series similar in form to ψ occur frequently in the sequel, we shall call them *triply odd series* inasmuch as they contain only odd powers of a and odd functions of odd multiples of τ.

§ 3. *The Equations of Variation.*

We wish to show the existence of periodic solutions of (4) which are expansible as power series in ε and which for $\varepsilon=0$ reduce to $x\equiv y\equiv 0$, $z=\psi$. Let us substitute in (4)

$$(t-t_0)=(\tau-\tau_0)\sqrt{\tfrac{1}{8}(1+\delta)},\quad z=\psi+w, \tag{6}$$

where w vanishes with ε. If we denote derivatives with respect to τ by dots, and expand the right members as power series in ε, x, y and w, then the differential equations (4) become

$$\left.\begin{aligned}
&\ddot{x}-2\sqrt{\tfrac{1}{8}(1+\delta)}\,\dot{y}-\tfrac{1}{8}(1+\delta)x=x\sum_{i=0}^{\infty}X_i^{(0)}+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}X_i^{(j)}\varepsilon^j,\\
&\ddot{y}+2\sqrt{\tfrac{1}{8}(1+\delta)}\,\dot{x}-\tfrac{1}{8}(1+\delta)y=y\sum_{i=0}^{\infty}Y_i^{(0)}+y\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}Y_i^{(j)}\varepsilon^j,\\
&w+W_0w=\sum_{i=2}^{\infty}W_i^{(0)}+\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}W_i^{(j)}\varepsilon^j.
\end{aligned}\right\} \tag{7}$$

Before characterizing the undefined terms in (7), we shall define a *triply even series*, as such series occur frequently in the sequel. A triply even series is a power series in a^2 with sums of cosines of even multiples of τ in the coefficients. The highest multiple of τ in the coefficient of a^{2k} is $2k$. Such a series is called triply even since it is even in a and only even functions of even multiples of τ enter the coefficients.

The $X_i^{(0)}$, $Y_i^{(0)}$ and $W_i^{(0)}$ are homogeneous polynomials of degree i in x, y and w, but only even powers of x, and y enter while w may enter to any power. The coefficients in $X_i^{(0)}$ and $Y_i^{(0)}$ are triply even or triply odd series, according as i is even or odd, respectively. The coefficients in $W_i^{(0)}$ are triply odd or triply even series, according as i is even or odd, respectively. As far as the computations have been carried out we have, *Proc.* (5),

$$
\begin{aligned}
W_0 &= 1-a^2\left(\frac{9}{2}-9\cos 2\tau\right)+a^4\left(\frac{687}{32}-48\cos 2\tau+\frac{177}{8}\cos 4\tau\right)+\ldots.,\\
X_0^{(0)} &= 2-3\,a^2(1-4\cos 2\tau)+\ldots.,\\
Y_0^{(0)} &= -1-\frac{3}{2}a^2(1+2\cos 2\tau)+\ldots.,\\
X_1^{(0)} &= -48(1+\delta)\,\psi w,\quad Y_1^{(0)}=12(1+\delta)\,\psi w,\\
X_2^{(0)} &= x^2[16-24\,a^2(7-10\cos 2\tau)+\ldots.]-y^2[24-36\,a^2(2-5\cos 2\tau)+\ldots.]\\
&\qquad -w^2[24-108\,a^2(4-5\cos 2\tau)+\ldots.],\\
Y_2^{(0)} &= -x^2[24-36\,a^2(2-5\cos 2\tau)+\ldots.]+y^2[6-3\,a^2(1-10\cos 2\tau)+\ldots.]\\
&\qquad +w^2[6-9a^2(7-10\cos 2\tau)+\ldots.],\\
W_2^{(0)} &= -x^2[24\,a\sin\tau+\ldots.]+y^2[6\,a\sin\tau+\ldots.]+w^2[18\,a\sin\tau+\ldots.].
\end{aligned}
$$

The $X_i^{(j)}$, $Y_i^{(j)}$ and $W_i^{(j)}$ for $j>0$ are likewise polynomials of degree i in x, y and w, but y enters only to even degrees. If j is odd, the $X_i^{(j)}$ are even in x, and $Y_i^{(j)}$ and $W_i^{(j)}$ are odd in x. If j is even, the $X_i^{(j)}$ are odd in x, and $Y_i^{(j)}$ and $W_i^{(j)}$ are even in x. The coefficients in $X_i^{(j)}$ and $Y_i^{(j)}$ are triply even or triply odd series, according as w enters to even or odd degrees, respectively, but in $W_i^{(j)}$ the coefficients are triply odd or triply even series, according as w enters to even or odd degrees, respectively. It is observed that $X_0^{(j)}=0$ if j is even, and $Y_0^{(j)}=W_0^{(j)}=0$ if j is odd.

The expansions in the right members in (7) are valid only in a certain region of convergence. It is seen from the values of r_1 and r_2 in equations (4) that x, y and z must satisfy the inequalities

$$
\begin{aligned}
-\tfrac{1}{4} &< +\varepsilon+\varepsilon^2-x-2\varepsilon x+x^2+y^2+z^2<\tfrac{1}{4},\\
-\tfrac{1}{4} &< -\varepsilon+\varepsilon^2+x-2\varepsilon x+x^2+y^2+z^2<\tfrac{1}{4}.
\end{aligned}
$$

The region of convergence is obtained by replacing the inequality signs with signs of equality. When referred to the fixed $\xi\eta$-axes the region is found to be the spindle formed by rotating the common portion of two intersecting circles about their common chord. These circles have their centres at $1/2+\varepsilon, 0, 0$ and $-1/2+\varepsilon, 0, 0$ and have radii $1/\sqrt{2}$.

The equations of variation are obtained by putting $\epsilon=0$ in (7) and taking only the linear terms in x, y and w in the right members. They are

$$\left.\begin{aligned}&\ddot{x}-2\sqrt{\{\tfrac{1}{8}(1+\delta)\}}\,\dot{y}-[\tfrac{1}{8}(1+\delta)+X_0^{(0)}]x=0,\\&\ddot{y}+2\sqrt{\{\tfrac{1}{8}(1+\delta)\}}\,\dot{x}-[\tfrac{1}{8}(1+\delta)+Y_0^{(0)}]y=0,\\&\ddot{w}+W_0 w=0.\end{aligned}\right\}\tag{8}$$

These equations are the same as *Proc.* (6) if we put $m_0=\frac{1}{2}$ in the latter. The solutions of the first two equations are obtained from *Proc.* (8) by putting $m_0=\frac{1}{2}$. They are

$$\left.\begin{aligned}x&=A_1 e^{\sqrt{-1}\beta_1\tau}u_1+A_2 e^{-\sqrt{-1}\beta_1\tau}u_2+A_3 e^{\beta_2\tau}u_3+A_4 e^{-\beta_2\tau}u_4,\\y&=\sqrt{-1}[A_1 e^{\sqrt{-1}\beta_1\tau}v_1-A_2 e^{-\sqrt{-1}\beta_1\tau}v_2]+A_3 e^{\beta_2\tau}v_3-A_4 e^{-\beta_2\tau}v_4,\end{aligned}\right\}\tag{9}$$

where A_i $(i=1,2,3,4)$ are the constants of integration, u_i and v_i $(i=1,2,3,4)$ are periodic functions of τ having the period 2π; and

$$\beta_j=\sum_{k=0}^{\infty}\beta_j^{(2k)}a^{2k}\quad(j=1,2)$$

are power series in a^2 with real constant coefficients which are determined by the conditions that u_i and v_i shall be periodic. The values of $-(\beta_1^{(0)})^2$ and $(\beta_2^{(0)})^2$ are the roots of the quadratic

$$64\gamma^2-48\gamma-119=0,$$

and, therefore,

$$\beta_1^{(0)}=[\sqrt{2}-\tfrac{3}{8}]^{\frac{1}{2}},\quad \beta_2^{(0)}=[\sqrt{2}+\tfrac{3}{8}]^{\frac{1}{2}}.$$

The u_i and v_i are power series in a^2 with sums of sines and cosines of even multiples of τ in the coefficients, the highest multiple of τ in the coefficient of a^{2k} being $2k$. In u_i and v_i $(i=1,2)$ the coefficients of the sines are purely imaginary and the coefficients of the cosines are real. In u_i and v_i $(i=3,4)$ all the coefficients are real. Further,

$$\begin{gathered}u_1(\sqrt{-1})=u_2(-\sqrt{-1}),\quad u_3(\tau)=u_4(-\tau),\\v_1(\sqrt{-1})=v_2(-\sqrt{-1}),\quad v_3(\tau)=v_4(-\tau),\\u_i(0)=1\ (i=1,2,3,4).\end{gathered}$$

The determinant of the solutions (9) and their first derivatives is a constant,* and at $\tau=0$ its value is

$$\Delta=10\sqrt{-119}+\text{terms in }a^2.\tag{10}$$

This determinant is different from zero for $a=0$ and will remain different from zero for a^2 sufficiently small. Hence, the solutions (9) constitute a fundamental set.

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* Moulton's "Periodic Orbits," § 18.

The general solution of the last equation of (9) is the same as *Proc.* (7), viz.,

$$\left.\begin{aligned} w &= A_5\phi + A_6[\chi + A\tau\phi], \\ \phi &= \cos\tau + \frac{9}{16}a^2[\cos 3\tau - \cos\tau] + \ldots, \\ \chi &= \sin\tau + \frac{9}{16}a^2[\sin 3\tau + 5\sin\tau] + \ldots, \\ A &= -\frac{9}{2}a^2 + \frac{141}{16}a^4 + \ldots, \end{aligned}\right\} \tag{11}$$

where A_5 and A_6 are the constants of integration.

§ 4. *Existence of Symmetrical Periodic Orbits.*

Let us choose as initial conditions of (7)

$$x(0) = \alpha_1,\ \dot{x}(0) = 0,\ y(0) = 0,\ \dot{y}(0) = \alpha_2,\ w(0) = 0,\ \dot{w}(0) = \alpha_3. \tag{12}$$

With these initial conditions it can be shown from the differential equations (7) that x is even in τ and y and w are odd in τ. Hence, sufficient conditions that the solutions shall be periodic with the period 2π are

$$\dot{x} = y = w = 0 \text{ at } \tau = \pi. \tag{13}$$

When the conditions (12) are imposed on the solutions (9) and (11), we find that $A_1 = A_2$, $A_3 = A_4$, $A_5 = 0$, and that A_1, A_3 and A_6 are linear functions of $\alpha_i (i = 1, 2, 3)$, vanishing with α_i. Now let us integrate equations (7) as power series in ε and α_i, or, which is more convenient, in ε, A_1, A_3 and A_6. In so far as the terms which are linear in A_1, A_3 and A_6 are concerned, we obtain

$$\left.\begin{aligned} x_{10} &= A_1[e^{\sqrt{-1}\beta_1\tau}u_1 + e^{-\sqrt{-1}\beta_1\tau}u_2] + A_3[e^{\beta_2\tau}u_3 + e^{-\beta_2\tau}u_4], \\ y_{10} &= \sqrt{-1}A_1[e^{\sqrt{-1}\beta_1\tau}v_1 - e^{-\sqrt{-1}\beta_1\tau}v_2] + A_3[e^{\beta_2\tau}v_3 - e^{-\beta_2\tau}v_4], \\ w_{10} &= A_6[\chi + A\tau\phi]. \end{aligned}\right\} \tag{14}$$

When the periodicity conditions (13) are imposed on these solutions, we obtain

$$\left.\begin{aligned} 0 &= A_1[\sqrt{-1}\beta_1 + \dot{u}_1(0)][e^{\sqrt{-1}\beta_1\pi} - e^{-\sqrt{-1}\beta_1\pi}] + A_3[\beta_2 + \dot{u}_3(0)][e^{\beta_2\pi} - e^{-\beta_2\pi}] \\ &\quad + \text{terms in } \varepsilon \text{ and higher degree terms in } A_1, A_3, A_6 \text{ and } \varepsilon, \\ 0 &= \sqrt{-1}v_1(0)A_1[e^{\sqrt{-1}\beta_1\pi} - e^{-\sqrt{-1}\beta_1\pi}] + v_3(0)A_3[e^{\beta_2\pi} - e^{-\beta_2\pi}] \\ &\quad + \text{higher degree terms in } A_1, A_3, A_6 \text{ and } \varepsilon, \\ 0 &= A_6A\pi + \text{higher degree terms in } A_1, A_3, A_6 \text{ and } \varepsilon. \end{aligned}\right\} \tag{15}$$

The determinant of the linear terms in A_1, A_3 and A_6 is

$$D = [e^{\sqrt{-1}\beta_1\pi} - e^{-\sqrt{-1}\beta_1\pi}][e^{\beta_2\pi} - e^{-\beta_2\pi}] \left\{v_3(0)[\sqrt{-1}\beta_1 + \dot{u}_1(0)] - \sqrt{-1}v_1(0)[\beta_2 + \dot{u}_3(0)]\right\}. \tag{16}$$

The last factor has the form

$$\frac{5\sqrt{-119}}{7\sqrt{2}} + \text{terms in } a^2,$$

and is different from zero for a^2 sufficiently small. The second factor can not vanish as β_2 is real. Then the determinant D can vanish only when β_1 is an integer. When β_1 is not an integer, D is not zero; and hence equations (15) can be solved uniquely for A_1, A_3 and A_6 as power series in ε which vanish with ε and converge for $|\varepsilon|$ sufficiently small. Therefore symmetrical periodic solutions of (7) exist when β_1 is not an integer, and they have the form

$$x = \sum_{j=1}^{\infty} x_j \varepsilon^j, \quad y = \sum_{j=1}^{\infty} y_j \varepsilon^j, \quad w = \sum_{j=1}^{\infty} w_j \varepsilon^j, \tag{17}$$

where each x_j, y_j and w_j is separately periodic with the period 2π in τ.

From the following considerations of the differential equations (7) we show that x and y are odd in ε, and w is even in ε. The right member of the first equation in (7) is odd in x and ε, considered together, and even in y. The right member of the second equation is even in x and ε, considered together, and also even in y. Let the solutions (17) be denoted by

$$x=x(\varepsilon), \quad y=y(\varepsilon), \quad w=w(\varepsilon).$$

If the signs of x, y and ε be changed, the differential equations remain unchanged, and therefore,

$$x=-x(-\varepsilon), \quad y=-y(-\varepsilon), \quad w=w(-\varepsilon)$$

are also solutions. Since the solutions as power series in ε are unique, then

$$x(\varepsilon)=-x(-\varepsilon), \quad y(\varepsilon)=-y(-\varepsilon), \quad w(\varepsilon)=w(-\varepsilon),$$

or, x and y are odd in ε and w is even in ε. Hence, the periodic solutions of (7) have the form

$$x = \sum_{j=0}^{\infty} x_{2j+1}\, \varepsilon^{2j+1}, \quad y = \sum_{j=0}^{\infty} y_{2j+1}\, \varepsilon^{2j+1}, \quad w = \sum_{j=1}^{\infty} w_{2j}\, \varepsilon^{2j}. \tag{18}$$

The same result would be obtained in the construction of the solutions if the forms (17) were used instead of (18).

§ 5. *Existence of Symmetrical Periodic Orbits when β_1 is an Integer.*

When β_1 is an integer, the determinant (16) vanishes and the terms in (15) of higher degree in ε, A_1, A_3 and A_6 must be considered in order to establish the existence of symmetrical periodic orbits. We take the same initial conditions (12), and as in the previous section we integrate (7) as power series in A_1, A_3, A_6 and ε. In addition to the linear terms already obtained in (14), we need the quadratic terms in A_1, A_3 and A_6, and the term in A_1^3 in so far as it enters x. If these terms are denoted by x_{11}, y_{11}, w_{11} and x_{30}

respectively, then all these terms except w_{11} can be obtained from *Proc.* § 6 by putting $m_0=1/2$, $\lambda=0$ in the solutions there denoted by the same notation. Thus,

$$\left.\begin{aligned}
x_{11} &= \frac{1}{\Delta} A_1 A_6 Q_1 \tau [e^{\sqrt{-1}\beta_1 \tau} u_1 - e^{-\sqrt{-1}\beta_1 \tau} u_2] \\
&\qquad + \frac{\sqrt{-1}}{\Delta} A_3 A_6 Q_2 \tau [e^{\beta_2 \tau} u_3 - e^{-\beta_2 \tau} u_4], \\
y_{11} &= \frac{\sqrt{-1}}{\Delta} A_1 A_6 Q_1 \tau [e^{\sqrt{-1}\beta_1 \tau} v_1 + e^{-\sqrt{-1}\beta_1 \tau} v_2] \\
&\qquad + \frac{\sqrt{-1}}{\Delta} A_3 A_6 Q_2 \tau [e^{\beta_2 \tau} v_3 + e^{-\beta_2 \tau} v_4], \\
x_{30} &= \frac{1}{\Delta} A_1^3 Q_3 \tau [e^{\sqrt{-1}\beta_1 \tau} u_1 - e^{-\sqrt{-1}\beta_1 \tau} u_2],
\end{aligned}\right\} \qquad (19)$$

where Q_1, Q_2 and Q_3 are power series in a^2 with real constant coefficients.

The term w_{11} is the solution of the differential equation

$$\ddot{w}_{11} + W_0 w_{11} = W_{11}, \qquad (20)$$

where the right member is a linear function of x_{10}^2, y_{10}^2, w_{10}^2 multiplied by a triply odd series. The complementary function of (20) is the same as the solution of the last equation of (8), and on using the method of the variation of parameters we obtain

$$w_{11} = \tau [A_1^2 \phi_1 + A_3^2 \phi_3] + A_6^2 [\text{non-periodic terms}] + \text{periodic terms},$$

where ϕ_1 and ϕ_3 are power series similar to ϕ in (11).

When the periodicity conditions (13) are imposed on the solutions

$$\begin{aligned}
x &= x_{10} + x_{11} + x_{30} + \ldots, \\
y &= y_{10} + y_{11} + \ldots, \\
w &= w_{10} + w_{11} + \ldots,
\end{aligned}$$

we obtain the equations

$$\left.\begin{aligned}
0 &= A_3 [\beta_2 + \dot{u}_3(0)] \left\{ e^{\beta_2 \pi} - e^{-\beta_2 \pi} + \frac{\sqrt{-1}\pi}{\Delta} A_6 Q_2 [e^{\beta_2 \pi} + e^{-\beta_2 \pi}] \right\} \\
&\qquad \pm \frac{2\pi}{\Delta} A_1 [\sqrt{-1}\beta_1 + \dot{u}_1(0)] [A_6 Q_1 + A_1^2 Q_3] + \text{terms in } \varepsilon \\
&\qquad \text{and cubic and higher degree terms } A_1, A_3 \text{ and } A_6, \\
0 &= \sqrt{-1} v_3(0) A_3 \left\{ e^{\beta_2 \pi} - e^{-\beta_2 \pi} + \frac{\pi}{\Delta} A_6 Q_2 [e^{\beta_2 \pi} + e^{-\beta_2 \pi}] \right\} \\
&\qquad + \frac{\sqrt{-1}\pi}{\Delta} A_6 \left\{ \pm 2 v_1(0) A_1 Q_1 + v_3(0) A_3 Q_2 [e^{\beta_2 \pi} + e^{-\beta_2 \pi}] \right\} \\
&\qquad + \text{terms in } \varepsilon \text{ and cubic and higher degree terms in } A_1, \\
&\qquad A_3 \text{ and } A_6, \\
0 &= \pi [A A_6 + A_1^2 \phi_1(\pi) + A_3^2 \phi_3(\pi)] + \text{terms in } \varepsilon \text{ and higher degree} \\
&\qquad \text{terms in } A_1, A_3 \text{ and } A_6.
\end{aligned}\right\} \qquad (21)$$

Where the double sign occurs the $+$ is to be taken if β_1 is an even integer, and the $-$ if it is an odd integer.

The last equation of (21) can be solved for A_6 as a power series in $A_1^{\frac{3}{2}}$, $A_3^{\frac{3}{2}}$ and ε which vanishes with A_1 and A_6 but not with ε. We shall refer to this solution as (21c). When it is substituted in the second equation of (21) we obtain an equation in A_1, A_3 and ε which vanishes with A_1 and A_3 but not with ε, the terms of lowest degree being A_1^3 and A_3. As the coefficient of A_3 in this equation is a power series in a^2 with additional terms in $1/a^2$, it will be different from zero, in general, and the equation can be solved for A_3 as a power series in A_1^3 and ε which vanishes with A_1 but not with ε. Denote this series for A_3 by (21b). After substituting (21c) and (21b) in the first equation of (21), we obtain an equation (21a) in A_1 and ε which vanishes with these terms and in which the lowest power of A_1 alone is A_1^3. The coefficient of A_1^3 in this equation is a power series in a^2 with additional terms in $1/a^2$ and will, in general, be different from zero. Hence, (21a) can be solved for A_1 as a power series in $\varepsilon^{\frac{1}{3}}$ which vanishes with ε and converges for $|\varepsilon|$ sufficiently small. There are three solutions for A_1, but only one is real, the other two being complex. When this power series for A_1 is substituted in (21b) and (21c), A_3 and A_6 are likewise power series in $\varepsilon^{\frac{1}{3}}$. Therefore, when β_1 is an integer, periodic solutions exist having the form

$$x=\sum_{j=1}^{\infty} x^{(j)}\varepsilon^{\frac{j}{3}},\quad y=\sum_{j=1}^{\infty} y^{(j)}\varepsilon^{\frac{j}{3}},\quad w=\sum_{j=1}^{\infty} w^{(j)}\varepsilon^{\frac{j}{3}} \tag{22}$$

where each $x^{(j)}$, $y^{(j)}$ and $w^{(j)}$ is separately periodic with the period 2π in τ. These solutions converge for ε sufficiently small numerically. There is only one set of real solutions, the other two sets being complex.

In the practical construction of the solutions it can be shown that x and y are odd in $\varepsilon^{\frac{1}{3}}$, and that w is even in $\varepsilon^{\frac{1}{3}}$. This property of the solutions can be deduced directly from the differential equations (7) by an argument similar to that used at the end of § 4 if we replace ε in that section by $\varepsilon^{\frac{1}{3}}$. Then we may consider the solutions to have the form

$$x=\sum_{j=0}^{\infty} x^{(2j+1)}\varepsilon^{\frac{2j+1}{3}},\quad y=\sum_{j=0}^{\infty} y^{(2j+1)}\varepsilon^{\frac{2j+1}{3}},\quad w=\sum_{j=1}^{\infty} w^{(2j)}\varepsilon^{\frac{2j}{3}}. \tag{23}$$

§ 6. *Proof that all the Periodic Orbits are Symmetrical.*

We shall now consider the existence of periodic orbits in which the infinitesimal body is projected from the xy-plane in a direction which may not be perpendicular to the x-axis, and from a point which may not lie on the x-axis. These orbits, if they exist, are called the general orbits. They have the initial values

$$x(0)=\alpha_1,\ \dot{x}(0)=\alpha_2,\ y(0)=\alpha_3,\ \dot{y}(0)=\alpha_4,\ w(0)=0,\ \dot{w}(0)=\alpha_5. \tag{24}$$

The initial value of w can be taken to be zero without loss of geometric generality since the infinitesimal body must cross the xy-plane if the motion is to be periodic, and hence the initial time can be chosen as the time when the infinitesimal body crosses the xy-plane. Sufficient conditions that solutions having the initial values (24) shall be periodic are

$$\left.\begin{array}{lll} x(2\pi)-x(0)=0, & y(2\pi)-y(0)=0, & w(2\pi)-w(0)=0, \\ \dot{x}(2\pi)-\dot{x}(0)=0, & \dot{y}(2\pi)-\dot{y}(0)=0, & \dot{w}(2\pi)-\dot{w}(0)=0. \end{array}\right\} \qquad (25)$$

We shall now show that one of these conditions, viz., $\dot{w}(2\pi)-\dot{w}(0)=0$, can be suppressed.

So far no use has been made of the integral (5). It is by means of this integral that we show that one of the conditions in (25) is redundant. When (5) is transformed by the substitutions (6), it takes the form

$$\dot{x}^2+\dot{y}^2+(\dot{\psi}+\dot{w})^2+F(x,\, y^2,\, w;\, \epsilon)=0, \qquad (26)$$

where F is a power series in x, y^2, w and ϵ having periodic coefficients. Let us make in (26) the usual substitutions

$$\left.\begin{array}{lll} x=x(0)+\bar{x}, & y=y(0)+\bar{y}, & w=0+\bar{w}, \\ \dot{x}=\dot{x}(0)+\dot{\bar{x}}, & y=\dot{y}(0)+\dot{\bar{y}}, & \dot{w}=\dot{w}(0)+\dot{\bar{w}}, \end{array}\right\} \qquad (27)$$

where $\bar{x}$, $\dot{\bar{x}}$, $\bar{y}$, $\dot{\bar{y}}$, $\bar{w}$ and $\dot{\bar{w}}$ vanish at $\tau=0$. We shall denote the equation resulting from substituting (27) in (26) by (26a). By putting $\tau=0$ we obtain from (26a) an equation (26b) connecting the terms of (26a) which are independent of $\bar{x}, \ldots, \dot{\bar{w}}$. On substituting (26b) in (26a) we obtain an equation

$$G(\bar{x},\, \dot{\bar{x}},\, \bar{y},\, \dot{\bar{y}},\, \bar{w},\, \dot{\bar{w}})=0, \qquad (28)$$

in which at $\tau=0$ or 2π there are no terms independent of the arguments indicated. The coefficient of $\dot{\bar{w}}(2\pi)$ in this equation is $2[a+\dot{w}(0)]$, and it vanishes only when $\dot{w}(0)=-a$. If $^{w}(0)=-a$ then $\dot{z}(0)=0$, since $z=\psi+w$ and $\dot{\psi}(0)=a$, and as $z(0)=0$ it follows that $z\equiv 0$. As we desire solutions which are not identically zero we must take $\dot{w}(0)=-a$. Therefore the coefficient of $\dot{\bar{w}}(2\pi)$ in (28) is not zero at $\tau=2\pi$, and by the theory of implicit functions this equation can be solved uniquely for $\dot{w}(2\pi)$ as a power series

$$\dot{\bar{w}}(2\pi)=H\{\bar{x}(2\pi),\, \dot{\bar{x}}(2\pi),\, \bar{y}(2\pi),\, \dot{\bar{y}}(2\pi),\, \bar{w}(2\pi)\}, \qquad (29)$$

which vanishes with the arguments indicated and converges for sufficiently small values of the moduli of these arguments. When the first five conditions of (25) are satisfied it follows from (29) that the arguments of H are all zero, and therefore $\dot{\bar{w}}(2\pi)=0$. Then, since $\dot{\bar{w}}(2\pi)=\dot{w}(2\pi)-\dot{w}(0)$, it follows that the condition $\dot{w}(2\pi)-\dot{w}(0)=0$ is a consequence of the other conditions in (25) and may be suppressed.

We shall now integrate equations (7) as power series in α_i $(i=1, \ldots, 5)$ and ε. The differential equations from which the linear terms are obtained are the same as the equations of variation, and consequently the solutions are the same as (9) and (11). When the initial conditions (24) are imposed on these solutions, we find that $A_5=0$ and that the remaining A_i are linear functions of α_i which vanish with α_i. Then, as in § 4, it is more convenient to integrate (7) as power series in A_i and ε. On imposing the necessary conditions of (25) upon (9) and (11) we obtain

$$\left.\begin{aligned} 0 &= A_1[e^{2\sqrt{-1}\beta_1\pi}-1]+A_2[e^{-2\sqrt{-1}\beta_1\pi}-1]+A_3[e^{2\beta_2\pi}-1] \\ &\qquad +A_4[e^{-2\beta_2\pi}-1]+\ldots, \\ 0 &= [\sqrt{-1}\beta_1+\dot{u}_1(0)]\left\{A_1[e^{2\sqrt{-1}\beta_1\pi}-1]-A_2[e^{-2\sqrt{-1}\beta_1\pi}-1]\right\} \\ &\qquad +[\beta_2 v_3(0)+\dot{v}_3(0)]\left\{A_3[e^{2\beta_2\pi}-1]-A_4[e^{-2\beta_2\pi}-1]\right\}+\ldots, \\ 0 &= \sqrt{-1}\,v_1(0)\left\{A_1[e^{2\sqrt{-1}\beta_1\pi}-1]-A_2[e^{-2\sqrt{-1}\beta_1\pi}-1]\right\} \\ &\qquad +v_3(0)\left\{A_3[e^{2\beta_2\pi}-1]-A_4[e^{-2\beta_2\pi}-1]\right\}+\ldots, \\ 0 &= [-\beta_1 v_1(0)+\sqrt{-1}\,\dot{v}_1(0)]\left\{A_1[e^{2\sqrt{-1}\beta_1\pi}-1]+A_2[e^{-2\sqrt{-1}\beta_1\pi}-1]\right\} \\ &\qquad +[\beta_2 v_2(0)+\dot{v}_3(0)]\left\{A_3[e^{2\beta_2\pi}-1]+A_4[e^{-2\beta_2\pi}-1]\right\}+\ldots, \\ 0 &= 2\pi A\,A_6\,\dot{\phi}(0)+\ldots\ldots \end{aligned}\right\} \quad (30)$$

The determinant of the coefficients of A_i is

$$2\pi\Delta A\dot{\phi}(0)\,[e^{2\sqrt{-1}\beta_1\pi}-1]\,[e^{-2\sqrt{-1}\beta_1\pi}-1]\,[e^{2\beta_2\pi}-1]\,[e^{-2\beta_2\pi}-1], \qquad (31)$$

and it vanishes only when β_1 is an integer since β_2 is real and Δ, A and $\dot{\phi}(0)$ are different from zero for a not zero but sufficiently small numerically. If β_1 is not an integer the determinant (31) is different from zero and equations (30) can be solved uniquely for A_i as power series in ε which vanish with ε and converge for $|\varepsilon|$ sufficiently small. Hence, when β_1 is not an integer, the general orbits exist uniquely and have the same form as the symmetrical orbits. Since the general orbits include the symmetrical and both classes of orbits are unique, then all the periodic orbits are symmetrical when β_1 is not an integer.

When β_1 is an integer the coefficients of A_1 and A_2 in the first four equations of (30) vanish and in order to discuss the existence of general orbits, it is necessary to consider the terms of higher degree in (30). In addition to the linear terms (9) we require, as in § 5, the quadratic terms in A_i $(i=1, 2, 3, 4, 6)$ which contain τ as a factor, and the terms of x containing τ as a factor and of the third degree in A_1 and A_2. These terms are found in the same way as the corresponding terms were determined in §5. They have the form

$$\left.\begin{aligned}
x_{11} &= \frac{1}{\Delta} A_6 Q_1 \tau [A_1 e^{\sqrt{-1}\beta_1\tau} u_1 - A_2 e^{-\sqrt{-1}\beta_1\tau} u_2] \\
&\qquad + \frac{\sqrt{-1}}{\Delta} A_6 Q_2 \tau [A_3 e^{\beta_2\tau} u_3 - A_4 e^{-\beta_2\tau} u_4], \\
y_{11} &= \frac{\sqrt{-1}}{\Delta} A_6 Q_1 \tau [A_1 e^{\sqrt{-1}\beta_1\tau} v_1 + A_2 e^{-\sqrt{-1}\beta_1\tau} v_2] \\
&\qquad + \frac{\sqrt{-1}}{\Delta} A_6 Q_2 \tau [A_3 e^{\beta_2\tau} v_3 + A_4 e^{-\beta_2\tau} v_4], \\
w_{11} &= \tau [A_1 A_2 \phi_1 + A_3 A_4 \phi_3] + A_6^2 [\text{non-periodic terms}] \\
&\qquad + \text{periodic terms}, \\
x_{30} &= \frac{1}{\Delta} A_1 A_2 Q_3 \tau [A_1 e^{\sqrt{-1}\beta_1\tau} u_1 - A_2 e^{\sqrt{-1}\beta_1\tau} u_2].
\end{aligned}\right\} \quad (32)$$

When the necessary periodicity conditions of (25) are imposed on equations (32), (9) and (11), we obtain

$$\left.\begin{aligned}
0 &= A_3[e^{2\beta_2\pi}-1] + A_4[e^{-2\beta_2\pi}-1] + \frac{2\pi}{\Delta} A_6 Q_1 [A_1 - A_2] \\
&\quad + \frac{2\sqrt{-1}\pi}{\Delta} A_6 Q_2 [A_3 e^{2\beta_2\pi} - A_4 e^{-2\beta_2\pi}] + \frac{2\pi}{\Delta} A_1 A_2 Q_3 [A_1 - A_2] \\
&\quad + \text{terms in } \epsilon \text{ and higher degree terms in } A_i, \\
0 &= [\beta_2 + \dot{u}_3(0)] \Big\{ A_3[e^{2\beta_2\pi}-1] - A_4[e^{-2\beta_2\pi}-1] \Big\} \\
&\quad + \frac{2\pi}{\Delta} [\sqrt{-1}\beta_1 + \dot{u}_1(0)] [A_6 Q_1 + A_1 A_2 Q_3] [A_1 + A_2] \\
&\quad + \frac{2\sqrt{-1}\pi}{\Delta} A_6 Q_2 [\beta_2 + \dot{u}_3(0)] [A_3 e^{2\beta_2\pi} + A_4 e^{-2\beta_2\pi}] \\
&\quad + \text{terms in } \epsilon \text{ and higher degree terms in } A_i, \\
0 &= v_3(0) \Big\{ A_3[e^{2\beta_2\pi}-1] - A_4[e^{-2\beta_2\pi}-1] \Big\} \\
&\quad + \frac{2\sqrt{-1}\pi}{\Delta} v_1(0) A_6 Q_1 [A_1 + A_2] \\
&\quad + \frac{2\sqrt{-1}\pi}{\Delta} v_3(0) A_6 Q_2 [A_3 e^{2\beta_2\pi} + A_4 e^{-2\beta_2\pi}] \\
&\quad + \text{terms in } \epsilon \text{ and higher degree terms in } A_i, \\
0 &= [\beta_2 v_3(0) + \dot{v}_3(0)] \Big\{ A_3[e^{2\beta_2\pi}-1] + A_4[e^{-2\beta_2\pi}-1] \Big\} \\
&\quad + \frac{2\sqrt{-1}\pi}{\Delta} A_6 Q_1 [\sqrt{-1}\beta_1 v_1(0) + \dot{v}_1(0)] [A_1 - A_2] \\
&\quad + \frac{2\sqrt{-1}\pi}{\Delta} A_6 Q_2 [\beta_2 v_3(0) + \dot{v}_3(0)] [A_3 e^{2\beta_2\pi} - A_4 e^{-2\beta_2\pi}] \\
&\quad + \text{terms in } \epsilon \text{ and higher degree terms in } A_i, \\
0 &= 2\pi A A_6 \dot{\phi}(0) + 2\pi [A_1 A_2 \phi_1(0) + A_3 A_4 \phi_3(0)] \\
&\quad + \text{terms in } \epsilon \text{ and higher degree terms in } A_i.
\end{aligned}\right\} \quad (33)$$

The last equation of (33) can be solved for A_6 as a power series in A_1, A_2, A_3, A_4 and ϵ, and the terms of lowest degree in A_i are $A_1 A_2$ and $A_3 A_4$. The determinant of the coefficients of A_3 and A_4 in the third and fourth equations is

$$2\, v_3(0)\,[\beta_2 v_3(0) + \dot{v}_3(0)]\,[e^{2\beta_2\pi} - 1]\,[e^{-2\beta_2\pi} - 1],$$

and it is different from zero for a^2 sufficiently small since β_2 is real. Hence these two equations, when A_6 has been eliminated by the last equation, can be solved for A_3 and A_4 as power series in A_1, A_2 and ϵ in which the terms of lowest degree in A_1 and A_2 are $A_1^2 A_2$ and $A_1 A_2^2$. When these solutions for A_3, A_4 and A_6 are substituted in the second equation, the terms independent of ϵ contain $A_1^2 A_2$ or $A_1 A_2^2$ as a factor. Hence this equation can be solved for A_2 as a power series in A_1 and ϵ, and the term of lowest degree in A_1 is linear in A_1. Finally, when the solutions for A_2, A_3, A_4 A_6 are substituted in the first equation, we obtain an equation in A_1 and ϵ in which the terms of lowest degree in A_1 is A_1^3. As in (25a) the coefficient of A_1^3 in this equation is a power series in a^2 with additional terms in $1/a^2$ and will, in general, be different from zero; therefore this equation can be solved for A_1^3 as a power series in $\epsilon^{\frac{1}{3}}$ which vanishes with ϵ and converges for $|\epsilon|$ sufficiently small. There are three solutions as in the symmetrical orbits, but only one solution is real, the other two being complex. Since the real solutions for the general orbits and the symmetrical orbits are both unique and since the general include the symmetrical orbits, all the periodic orbits are symmetrical when β_1 is an integer. Consequently all the periodic orbits are symmetrical whether β_1 is an integer or not.

§ 7. *Constructions of the Solutions when β_1 is not an Integer.*

Let us substitute (18) in (7) and equate the coefficients of the various powers of ϵ. We obtain a series of differential equations which can be integrated step by step and the constants of integration arising at each step can be determined, as we shall show, so that the solutions shall be periodic and shall satisfy the symmetrical initial conditions

$$x = y = w = 0 \text{ at } \tau = 0.$$

When these initial conditions are imposed on the solutions (18) we obtain

$$\left.\begin{aligned} \dot{x}_{2j+1}(0) = y_{2j+1}(0) = 0, &\qquad (j = 0, \ldots . \infty), \\ w_{2j}(0) = 0, &\qquad (j = 1, \ldots . \infty). \end{aligned}\right\} \tag{34}$$

The differential equations for the terms in ϵ are

$$\left.\begin{aligned} x_1 - 2\sqrt{\tfrac{1}{8}(1+\delta)}\,\dot{y}_1 - [\tfrac{1}{8}(1+\delta) + X_0^{(0)}]\,x_1 = X_0^{(1)}, \\ \ddot{y}_1 + 2\sqrt{\tfrac{1}{8}(1+\delta)}\,\dot{x}_1 - [\tfrac{1}{8}(1+\delta) + Y_0^{(0)}]\,y_1 = 0, \end{aligned}\right\}$$

where $X_0^{(1)}$ is the same even power series as that represented in (7) by the same notation. The complementary functions of (35) are the same as (9), and the particular integrals can be obtained by the method of the variation of parameters. The complete solutions are thus found to be

$$\left.\begin{aligned} x_1&=A_1^{(1)}e^{\sqrt{-1}\beta_1\tau}u_1+A_2^{(1)}e^{-\sqrt{-1}\beta_1\tau}u_2+A_3^{(1)}e^{\beta_2\tau}u_3+A_4^{(1)}e^{-\beta_2\tau}u_4+C_1(\tau),\\ y_1&=\sqrt{-1}\,[A_1^{(1)}e^{\sqrt{-1}\beta_1\tau}v_1-A_2^{(1)}e^{-\sqrt{-1}\beta_1\tau}v_2]+A_3^{(1)}e^{\beta_2\tau}v_3-A_4^{(1)}e^{-\beta_2\tau}v_4+S_1(\tau),\end{aligned}\right\}\quad(35)$$

where $A_i^{(1)}(i=1, 2, 3, 4)$ are the constants of integration, and $C_1(\tau)$ and $S_1(\tau)$ are triply even and triply odd series, respectively. As such series occur frequently in the construction, we shall denote them by $C_j(\tau)$ and $S_j(\tau)$, respectively. Since β_1 is not an integer in this case and β_2 is real, the terms of the complementary function must be suppressed in (35) by choosing $A_i^{(1)}=0$, $(i=1, 2, 3, 4)$, and therefore the desired solutions become

$$x_1=C_1(\tau),\quad y_1=S_1(\tau). \tag{36}$$

The differential equation for the terms in ϵ^2 is

$$\ddot{w}_2+W_0w_2=W_2,$$

where W_2 is a triply odd series. The complete solution of this equation is obtained by employing the method of the variation of parameters, and it is found to be

$$w_2=B_1^{(2)}\phi+B_2^{(2)}[\chi+A\tau\phi]-ap_2\tau\phi+S_2(\tau), \tag{37}$$

where $B_1^{(2)}$ and $B_2^{(2)}$ are constants of integration, and p_2 is a power series in a^2 with real constant coefficients. In order to satisfy the periodicity and the initial conditions (34), the constants of integration must have the values

$$B_1^{(2)}=0,\quad B_2^{(2)}=\frac{ap_2}{A}, \tag{38}$$

in which case the solution (37) becomes

$$w_2=\frac{1}{a^2}\bar{S}_2(\tau), \tag{39}$$

where $\bar{S}_2(\tau)$ is a triply odd power series.

The remaining steps of the integration can be carried on in the same way. The solutions for x_{2j-1} and y_{2j-1} are obtained from the particular integrals alone since β_2 is real and β_1 is assumed to be a number which is not an integer. These particular integrals are triply even and triply odd series, respectively, except for a factor $a^{-2(j-2)}$ which is introduced through the factor a^{-2} in (39). The general solution for the w_{2j} is similar to (37) and it can be made periodic

by a proper choice of the constants of integration as in (38). At the steps $2j-1$ and $2j$ the solutions are

$$\left.\begin{aligned} x_{2j-1} &= \frac{1}{a^{2j-4}} C_{2j-1}(\tau), \\ y_{2j-1} &= \frac{1}{a^{2j-4}} S_{2j-1}(\tau), \\ w_{2j} &= \frac{1}{a^{2j}} S_{2j}(\tau). \end{aligned}\right\} \tag{40}$$

§ 8. *Construction of the Solutions when β_1 is an Integer.*

Let us substitute equations (23) in (7) and equate the coefficients of the various powers of $\epsilon^{\frac{1}{2}}$. As in the previous section we obtain a series of differential equations which can be integrated step by step, and the constants of integration can be determined, as we shall show, so that the solutions shall be periodic and shall satisfy the symmetrical initial conditions

$$\dot{x}(0) = y(0) = w(0) = 0.$$

When these conditions are imposed upon (23) we obtain

$$\left.\begin{aligned} \dot{x}^{(2j+1)}(0) = y^{(2j+1)}(0) &= 0, \qquad (j=0, \ldots . \infty), \\ w^{(2j)}(0) &= 0, \qquad (j=1, \ldots . \infty). \end{aligned}\right\} \tag{41}$$

The differential equations for the terms in $\epsilon^{\frac{1}{2}}$ are the same as the first two equations of (8) if we use the superscript 1 on x and y. Since β_1 is assumed to be an integer, the solutions of these equations which are periodic and which satisfy (41) are

$$\left.\begin{aligned} x^{(1)} &= A_1^{(1)}[e^{\sqrt{-1}\beta_1\tau}u_1 + e^{-\sqrt{-1}\beta_1\tau}u_2], \\ y^{(1)} &= \sqrt{-1}A_1^{(1)}[e^{\sqrt{-1}\beta_1\tau}v_1 - e^{-\sqrt{-1}\beta_1\tau}v_2], \end{aligned}\right\} \tag{42}$$

where $A_1^{(1)}$ is a constant which is undetermined at this step. The solutions (42) are real if $A_1^{(1)}$ is real.

The differential equation for the terms in $\epsilon^{\frac{2}{2}}$ is

$$\ddot{w}^{(2)} + W_0 w^{(2)} = W^{(2)} = (A_1^{(1)})^2 S^{(2)}(\tau). \tag{43}$$

As functions similar to $S^{(2)}(\tau)$ occur frequently in the construction of the remaining solutions, we shall define the function $S^{2i}(\tau)$ to be a homogeneous polynomial of degree $2i$ in the exponentials $e^{\sqrt{-1}\beta_1\tau}$ and $e^{-\sqrt{-1}\beta_1\tau}$ of which the coefficients are power series in odd powers of a with sums of sines and $\sqrt{-1}$ times cosines of odd multiples of τ in the coefficients. The highest multiple of τ in the coefficient of a^{2k+1} is $2k+1$. The coefficients of $[e^{\sqrt{-1}\beta_1\tau}]^j\,[e^{-\sqrt{-1}\beta_1\tau}]^k$ and

$[e^{\sqrt{-1}\beta_1\tau}]^k[e^{-\sqrt{-1}\beta_1\tau}]^j$, $j+k=2i$, differ only in the sign of $\sqrt{-1}$. The part of $S^{(2i)}(\tau)$ which is independent of the exponentials is a triply odd power series. If the exponentials are expressed in trigonometric form, then

$$S^{(2i)}(\tau)=\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{h=0}^{i}a^{2j+1}S_{h,j,l}^{(2i)}\sin\{2h\beta_1\pm(2l+1)\}\tau,$$

where $S_{h,j,l}^{(2i)}$ are real constants: that is, $S^{(2i)}(\tau)$ is the same as a triply odd power series except that the highest multiple of τ in the coefficient of a^{2j+1} is not $2j+1$ but $2(j+i\beta_1)+1$.

The general solution of (43) is found in precisely the same way as (37) was obtained. It has the form

$$w^{(2)}=B_1^{(2)}\phi+B_2^{(2)}[\chi+A\tau\phi]-a(A_1^{(1)})^2p^{(2)}\tau\phi+(A_1^{(1)})^2S_0^{(2)}(\tau), \tag{44}$$

where $p^{(2)}$ is a power series in a^2 with real constant coefficients and $S_0^{(3)}(\tau)$ is similar to $S^{(2)}(\tau)$. On imposing the periodicity and the initial conditions (41), we have

$$B_1^{(2)}=0,\quad B_2^{(2)}=\frac{a}{A}(A_1^{(1)})^2p^{(2)}. \tag{45}$$

Then the desired solution of (43) takes the form

$$w^{(2)}=\frac{1}{a^2}(A_1^{(1)})^2\overline{S}^{(2)}(\tau),$$

where $\overline{S}^{(2)}(\tau)$ is similar to $S^{(2)}(\tau)$. Thus $A_1^{(1)}$ remains undetermined at this step also, and the terms in ε^3 must be considered.

The left members of the differential equations for the terms in ε^3 are the same as the first two equations in (8) if we use the superscript 3 on x and y. Let us denote the right members by $X^{(3)}$ and $Y^{(3)}$, respectively. Then

$$X^{(3)}=(A_1^{(1)})^3\rho^{(3)}+X_0^{(1)},\quad Y^{(3)}=\sqrt{-1}(A_1^{(1)})^3\sigma^{(3)}. \tag{46}$$

The function $X_0^{(1)}$ is the same triply even power series as that denoted in (7) by the same notation. The functions $\rho^{(2i+1)}$ and $\sigma^{(2i+1)}$ are homogeneous polynomials of degree $2i+1$ in the exponentials $e^{\sqrt{-1}\beta_1\tau}$ and $e^{-\sqrt{-1}\beta_1\tau}$, and the coefficients are power series similar in form to u_1 and u_2 in (9). In $\rho^{(2i+1)}$ the coefficients of $[e^{\sqrt{-1}\beta_1\tau}]^j[e^{-\sqrt{-1}\beta_1\tau}]^k$ and $[e^{\sqrt{-1}\beta_1\tau}]^k[e^{-\sqrt{-1}\beta_1\tau}]^j$, $j+k=2i+1$, differ only in the sign of $\sqrt{-1}$; while in $\sigma^{(2i+1)}$ the coefficients of $[e^{\sqrt{-1}\beta_1\tau}]^j[-e^{-\sqrt{-1}\beta_1\tau}]^k$ and $[e^{\sqrt{-1}\beta_1\tau}]^k[-e^{-\sqrt{-1}\beta_1\tau}]^j$ differ only in the sign of $\sqrt{-1}$.

The complementary functions of the differential equations in $x^{(3)}$ and $y^{(3)}$ are the same as (9). Since $X^{(3)}$ and $Y^{(3)}$ contain terms which have exactly the same period as the periodic parts of the complementary functions, the par-

ticular integrals will contain non-periodic terms. By the method of the variation of parameters we find that the general solutions for $x^{(3)}$ and $y^{(3)}$ are

$$\left.\begin{aligned} x^{(3)} &= A_1^{(3)} e^{\sqrt{-1}\beta_1\tau} u_1 + A_2^{(3)} e^{-\sqrt{-1}\beta_1\tau} u_2 + A_3^{(3)} e^{\beta_2\tau} u_3 + A_4^{(3)} e^{-\beta_2\tau} u_4 \\ &\quad + \tau[(A_1^{(1)})^3 P_1^{(3)} + P_2^{(3)}][e^{\sqrt{-1}\beta_1\tau} u_1 - e^{-\sqrt{-1}\beta_1\tau} u_2] + \bar{\rho}^{(3)}, \\ y^{(3)} &= \sqrt{-1}[A_1^{(3)} e^{\sqrt{-1}\beta_1\tau} v_1 - A_2^{(3)} e^{-\sqrt{-1}\beta_1\tau} v_2] + A_3^{(3)} e^{\beta_2\tau} v_3 - A_4^{(3)} e^{-\beta_2\tau} v_4 \\ &\quad + \sqrt{-1}\tau[(A_1^{(1)})^3 P_1^{(3)} + P_2^{(3)}][e^{\sqrt{-1}\beta_1\tau} v_1 + e^{-\sqrt{-1}\beta_1\tau} v_2] + \sqrt{-1}\bar{\sigma}^{(3)}, \end{aligned}\right\} \quad (47)$$

where $A_i^{(3)}$ $(i=1, 2, 3, 4)$ are the constants of integration, $P_i^{(3)}$ $(i=1, 2)$ are power series in a^2 with constant coefficients, and $\bar{\rho}^{(3)}$ and $\bar{\sigma}^{(3)}$ are similar to $\rho^{(3)}$ and $\sigma^{(3)}$, respectively. In order that the solutions shall be periodic, then

$$A_3^{(3)} = A_4^{(3)} = 0, \quad P_1^{(3)}(A_1^{(1)})^3 + P_2^{(3)} = 0.$$

The solutions for $A_1^{(1)}$ are

$$A_1^{(1)} = P_1^{(1)}, \quad \omega P_1^{(1)} \text{ or } \omega^2 P_1^{(1)},$$

where $P_1^{(1)}$ is a power series in a^2 with real constant coefficients, and ω, ω^2 are the imaginary cube roots of unity. As the imaginary solutions for $A_1^{(1)}$ lead to imaginary orbits, we retain only the real solution. Then, on imposing the initial conditions (41) on (47) we find that $A_1^{(3)} = A_2^{(3)}$, and the desired solutions for $x^{(3)}$ and $y^{(3)}$ become

$$x^{(3)} = A_1^{(3)}[e^{\sqrt{-1}\beta_1\tau} u_1 + e^{-\sqrt{-1}\beta_1\tau} u_2] + \bar{\rho}^{(3)},$$
$$y^{(3)} = \sqrt{-1} A_1^{(3)}[e^{\sqrt{-1}\beta_1\tau} v_1 - e^{-\sqrt{-1}\beta_1\tau} v_2] + \sqrt{-1}\bar{\sigma}^{(3)}.$$

The constant $A_1^{(3)}$ remains arbitrary at this step and the terms up to $\epsilon^{\frac{5}{2}}$ must be considered before it can be determined. The solutions for $x^{(3)}$ and $y^{(3)}$ are real provided $A_1^{(3)}$ is real, which is found to be the case.

The general solutions of the differential equation in $\epsilon^{\frac{4}{2}}$ is obtained in the same way as (44) was found, and it can be made to satisfy the periodicity and initial conditions by a choice of the constants of integration as in (45). When these conditions have been imposed, the solution for $w^{(4)}$ takes the form

$$w^{(4)} = \frac{1}{a^4}[A_1^{(3)} S^{(2)} + S^{(4)}].$$

The differential equations in $\epsilon^{\frac{5}{2}}$ are the same as (8) in the left members if we use the superscripts 5. If we denote the right members by $X^{(5)}$ and $Y^{(5)}$, respectively, then

$$X^{(5)} = \frac{1}{a^2}[A_1^{(3)}\rho_5^{(3)} + \rho_5^{(5)}], \quad Y^{(5)} = \frac{\sqrt{-1}}{a^2}[A_1^{(3)}\sigma_5^{(3)} + \sigma_5^{(5)}],$$

where $\rho_5^{(3)}$, $\rho_5^{(5)}$ and $\sigma_5^{(3)}$, $\sigma_5^{(5)}$ are similar to $\rho^{(2k+1)}$ and $\sigma^{(2k+1)}$, respectively. The

complete solutions of the equations in $x^{(5)}$ and $y^{(5)}$ are readily found in the same way as (47) were obtained. They are:

$$
\left.\begin{aligned}
x^{(5)} &= A_1^{(5)} e^{\sqrt{-1}\beta_1\tau} u_1 + A_2^{(5)} e^{-\sqrt{-1}\beta_1\tau} u_2 + A_3^{(5)} e^{\beta_2\tau} u_3 + A_4^{(5)} e^{-\beta_2\tau} u_4 \\
&\quad + \frac{\tau}{a^2}\,[P_1^{(5)} A_1^{(3)} + P_2^{(5)}]\,[e^{\sqrt{-1}\beta_1\tau} u_1 - e^{-\sqrt{-1}\beta_1\tau} u_2] + \frac{1}{a^2}\rho^{(5)}, \\
y^{(5)} &= \sqrt{-1}\,[A_1^{(5)} e^{\sqrt{-1}\beta_1\tau} v_1 - A_2^{(5)} e^{-\sqrt{-1}\beta_1\tau} v_2] + A_3^{(5)} e^{\beta_2\tau} v_3 - A_4^{(5)} e^{-\beta_2\tau} v_4 \\
&\quad + \frac{\sqrt{-1}\,\tau}{a^2}\,[P_1^{(5)} A_1^{(3)} + P_2^{(5)}]\,[e^{\sqrt{-1}\beta_1\tau} v_1 + e^{-\sqrt{-1}\beta_1\tau} v_2] + \frac{\sqrt{-1}}{a^2}\sigma^{(5)},
\end{aligned}\right\} \quad (48)
$$

where $P_i^{(5)}$ $(i=1,\ 2)$ are power series in a^2 with real constant coefficients. The solutions for $x^{(5)}$ and $y^{(5)}$ can be made periodic by putting $A_3^{(5)} = A_4^{(5)} = 0$, and then choosing $A_1^{(3)}$ so that

$$P_1^{(5)} A_1^{(3)} + P_2^{(5)} = 0,$$

from which it follows that $A_1^{(3)}$ is real. From the initial conditions (41) we have $A_1^{(5)} = A_2^{(5)}$.

The remaining steps of the integration can be carried on in the same way. The solution at the step $2j$ can be made to satisfy the periodicity and the initial conditions by a proper choice of the constants of integration arising at that step. The solution for w_{2j} contains the constant $A_1^{(2j-1)}$ which is not determined until the step $2j+1$. The non-periodic parts of the complete solutions at the step $2j+1$ are similar to those in (48), and $A_1^{(2j-1)}$ enters these solutions with the same coefficient as $A_1^{(3)}$ enters (48) except for the factor $1/a^{2j-2}$. The solutions can be made to satisfy the initial and periodicity conditions by putting

$$A_1^{(2j+1)} = A_2^{(2j+1)}, \quad A_3^{(2j+1)} = A_4^{(2j+1)} = 0,$$

and then determine the $A_1^{(2j-1)}$ so that the coefficient of τ shall be zero. The desired solutions at the steps $2j$ and $2j+1$ are

$$
\begin{aligned}
w^{(2j)} &= \frac{1}{a^{2j}} S^{2j}(\tau), \\
x^{(2j+1)} &= A_1^{(2j+1)}\,[e^{\sqrt{-1}\beta_1\tau} u_1 + e^{-\sqrt{-1}\beta_1\tau} u_2] + \frac{1}{a^{2j-2}}\rho^{(2j+1)}, \\
y^{(2j+1)} &= \sqrt{-1} A_1^{(2j+1)}\,[e^{\sqrt{-1}\beta_1\tau} v_1 - e^{-\sqrt{-1}\beta_1\tau} v_2] + \frac{\sqrt{-1}}{a^{2j-2}}\sigma^{(2j+1)}.
\end{aligned}
$$

Thus the integration can be carried on to any desired degree of accuracy.

The solutions of (7), whether as power series in ε or $\varepsilon^{\frac{1}{2}}$, must satisfy the integral (5) identically in ε or $\varepsilon^{\frac{1}{2}}$. Thus this integral, besides being of use in proving that all the periodic orbits are symmetrical, serves as a check on the computations.

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The Finite Groups of Birational Transformations of a Net of Cubics.

BY LEWIS C. COX.

Introduction.

1. The classification of non-linear periodic birational transformations in the plane into finite groups is due to S. Kantor * and A. Wiman.† The former showed that all periodic birational transformations in a plane can be transformed by combinations of quadratic transformations into a finite number of types having at most eight fundamental points. In the case of seven fundamental points, he determined the different transformations, but in their groups he made errors which A. Wiman subsequently corrected.

The method of Wiman was dependent upon the fact that these groups are isomorphic with the groups of transformations of the bitangents of a plane quartic C_4 which were known.‡

The object of this paper is to establish a method which enables one to determine the Cremona transformations with seven fundamental points which correspond to a given linear transformation of the quartic curve.

The method used is to first find the twenty-eight bitangents of the quartic. Hence a cubic surface is determined, having the quartic as a plane section of a particular cone which is tangent to the surface and has its vertex on the surface. This cubic surface is then depicted upon the plane of the seven fundamental points. The space transformations leaving the surface invariant and corresponding to a collineation of the quartic are next found. The corre-

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* S. Kantor, "Premiers Fondaments Pour Une Theorie Des Transformations Periodiques Univoques," *Atti della Reale Accademia delle scienze fisiche e matematiche di Napoli*, Series 2, Vol. III, No. 7 and Vol. IV, No. 2 (1891), pp. 1–335.

† A. Wiman, "Ueber die Endlichen Gruppen von eindeutigen Transformationen in der Ebene," *Mathematische Annalen*, Vol. XLVIII (1896); pp. 194–240.

‡ R. de Paolis, "Le trasformazioni piane doppie," *Atti delle Reale Accademia di Lincei*, Series 3a, Vol. I (1877), pp. 511–544. "Le trasformazioni piane doppia di secondo ordine, e la sua applicazione alla geometria non euclidea," *Atti d. R. Accad. di Lincei*, Series 3a, Vol. II (1878), pp. 31–50. "La trasformazione piana doppia di terzo ordine, primo genere, e la sua applicazione alle curve del quarto ordine," *Atti d. R. Accad. di Lincei*, Series 3a, Vol. II (1878), pp. 851–878.

sponding transformation on the plane of the seven fundamental points can then be determined. The method of determination is first developed for any collineation and is given in full for the cases in which the quartic is invariant under the groups of collineations G_{24}, G_{96}, G_{168}.

General Case.

2. Consider the cubic surface

$$F_3 \equiv f_1(x, y, z)u^2+2f_2(x, y, z)u+f_3(x, y, z)=0, \tag{1}$$

in which $f_i(x, y, z)$ is a ternary form of order i. Equation (1) can be arranged in the form

$$\{f_1(x, y, z)u+f_2(x, y, z)\}^2=f_2^2(x, y, z)-f_1(x, y, z)f_3(x, y, z), \tag{2}$$

which represents a composite surface made up of F_3 and the plane $f_1(x,y,z)=0$, the latter being tangent to F_3 at the point $0\equiv(0,0,0,1)$. Solving for u we get

$$u=\frac{-f_2(x, y, z)\pm\sqrt{f_2^2(x, y, z)-f_1(x, y, z)f_3(x, y, z)}}{f_1(x, y, z)}. \tag{3}$$

Hence, any line

$$\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$$

through the point $0\equiv(0,0,0,1)$ on the surface meets F_3 in two other points. These two residual points will coincide if

$$f_2^2(\lambda, \mu, \nu)-f_1(\lambda, \mu, \nu)f_3(\lambda, \mu, \nu)=0,$$

that is, the line is tangent to the surface and passes simply through 0.

Therefore,

$$K_4\equiv f_2^2(x, y, z)-f_1(x, y, z)f_3(x, y, z)=0, \tag{4}$$

is the tangent cone with its vertex at 0. The plane $u=0$ upon which we project $F_3=0$ is a double plane because the residual intersections of the lines through 0 with the surface $F_3=0$ project into single points. The cone $K_4=0$ intersects the plane $u=0$ in the quartic curve

$$C_4\equiv\left\{\begin{matrix} f_2^2-f_1f_3=0, \\ u\ =0. \end{matrix}\right\} \tag{5}$$

From the form of C_4 it is seen that $\left\{\begin{matrix} f_1(x, y, z)=0 \\ u\ =0 \end{matrix}\right\}$ is a bitangent to C_4.

The Lines on F_3.

3. Let $\phi_1(x, y, z)=0$ be the equation of the plane through 0 and one of the other twenty-seven bitangents to C_4. If one of the variables is eliminated

between $\begin{Bmatrix}\phi_1=0\\ u=0\end{Bmatrix}$ and (5) the remaining binary form must be a perfect square, whose roots determine the coordinates of the points of contact. Hence, eliminating z between

$$\phi_1(x,y,z)=0 \quad\text{and}\quad (f_1u+f_2)^2=f_2^2-f_1f_3$$

we obtain an expression that is rationally factorable:

$$[F_1u+F_2+\phi_2]\,[F_1u+F_2-\phi_2]=0, \tag{6}$$

in which F_1, F_2 and ϕ_2 are binary forms in x and y. Since equation (2) represents a composite surface whose components are F_3 and $f_1=0$, the equations

$$\begin{Bmatrix}[F_1u+F_2+\phi_2]\,[F_1u+F_2-\phi_2]=0\\ \phi_1(x,y,z)\quad=0\end{Bmatrix} \tag{7}$$

represent the curves $\begin{Bmatrix}\phi_1(x,y,z)=0\\ F_3\quad=0\end{Bmatrix}$ and $\begin{Bmatrix}f_1(x,y,z)=0\\ \phi_1(x,y,z)=0\end{Bmatrix}$, the latter defining an extraneous line not lying on F_3. Eliminating the factor f_1 from the factorable equation in (7) we find a remaining linear factor. This taken simultaneously with $\phi_1(x,y,z)=0$ fixes a line of F_3. The residual section of $\begin{Bmatrix}\phi_1(x,y,z)=0\\ F_3\quad=0\end{Bmatrix}$ is a proper conic section.

Therefore, we conclude that the planes through 0 and the twenty-seven bitangents to C_4 meet F_3 in the twenty-seven lines on F_3 and in twenty-seven residual conic sections. Hence each part of the composite curve, consisting of a line and its residual conic which lie not only on F_3 but also in the plane through 0 and one of the twenty-seven bitangents, corresponds to one of the twenty-seven bitangents to C_4.

The plane $f_1(x,y,z)=0$ is tangent to F_3 at 0. It intersects F_3 in the cubic curve

$$\begin{Bmatrix}f_1(x,y,z)=0,\\ 2f_2(x,y,z)u+f_3(x,y,z)=0.\end{Bmatrix} \tag{8}$$

It also intersects the plane $u=0$ in

$$\begin{Bmatrix}f_1(x,y,z)=0,\\ u\quad=0,\end{Bmatrix}$$

which is the twenty-eighth bitangent to C_4. Therefore, the point 0 and the cubic curve (8) correspond to the twenty-eighth bitangent to C_4.

The question arises, which sign in the second member of the equation

$$F_1u+F_2=\pm\phi_2 \tag{9}$$

corresponds to the cubic curve. Consider the equation

$$uf_1+f_2=+\sqrt{f_2^2-f_1f_3} \tag{10}$$

which is found from (2). Substitute the coordinates of a point $P\equiv(x, y, z, u)$ which lies on the cubic (8). The equation (10) is satisfied. Hence, the plus sign in the second member of (9) corresponds to the cubic (8). The point (0, 0, 0, 1) is the only point satisfying

$$uf_1+f_2=-\sqrt{f_2^2-f_1f_3}.$$

Hence, the minus sign corresponds to the point 0.

Correspondence of Transformations.

4. When C_4 is of genus 3, the only transformations which leave it invariant are collineations. The quartic curves admitting groups of such transformations are enumerated by S. Kantor.*

Let T represent any linear transformation which leaves C_4 invariant. It will permute the bitangents among themselves. From equation (2) we find

$$f_1(x, y, z)u+f_2(x, y, z)=\pm\{f_1(x', y', z')u'+f_2(x', y', z')\}. \qquad (11)$$

The relations (11) and T define two space transformations Q', Q''. Any plane through 0 and a bitangent to C_4 is changed by Q' or Q'' into a plane through 0 and some bitangent. Hence, the lines in which these planes meet F_3 are either interchanged, in which case the space transformation on F_3 is linear, or else each line is changed into the residual conic in the plane of the second line and the transformation is quadratic.

If we now use the Grassman depiction of the cubic surface F_3 upon a second plane $w=0$, the plane sections through 0 are transformed into the ∞^2 cubics through the image of 0 and having for additional basis points the images of the six special lines on F_3 which are depicted as the fundamental points. The point 0, the cubic curve (8) and the composite curves lying on F_3 in the twenty-seven planes through 0 and the lines of the surface are depicted in the plane $w=0$ as the complete fundamental system of the required Cremona transformation.

Since Q', Q'' interchange the twenty-eight special sections of F_3, one changing a line into a line, the other changing a line into a conic, the corresponding transformations T', T'' interchange the parts of the fundamental system. Hence, from T we can derive two birational Cremona transformations.

The Geiser Transformation.

5. In considering $T\equiv\begin{pmatrix} x, & y, & z \\ x', & y', & z' \end{pmatrix}$ in the plane $u=0$ we are led to two

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* S. Kantor, "Theorie der Endlichen Gruppen von eindeutigen Transformationen in der Ebene," Berlin (1895); 120 pages, see p. 86.

space transformations defined by the equations of T and (11). They are:

$$I \equiv \begin{pmatrix} x, & y, & z, & u \\ x', & y', & z', & u' \end{pmatrix},$$

and

$$\Gamma = \begin{cases} x = x' \\ y = y' \\ z = z' \\ w = -\{u'f_1(x', y', z') + f_2(x', y', z')\} - f_2(x', y', z'). \end{cases}$$

The former is the identical transformation.

The latter is a non-linear transformation possessing the following properties. It leaves all the planes and lines through 0 invariant, but interchanges the points P' and P'' which lie on F_3 and are collinear with 0. Therefore, it changes any line on F_3 into the residual conic lying on F_3 and coplanar with 0 and the line. Hence, the transformation is quadratic. The transformation Γ leaves invariant the locus of points R on F_3 which are fixed by the condition

$$f_2^2(\lambda, \mu, \nu) - f_1(\lambda, \mu, \nu) f_3(\lambda, \mu, \nu) = 0.$$

The cone $K_4 = 0$ and the surface $F_3 = 0$ have contact along a sextic space curve C_6 which is invariant point for point under the transformations I and Γ.

Hence, the transformation θ_2 in the plane $w = 0$ corresponding to Γ has the following properties: It interchanges the images of P' and P''. Therefore θ_2 is involutorial. It leaves invariant as a whole every cubic curve through the seven fundamental points. It leaves invariant point for point the image of C_6, which is a plane sextic curve Γ_6 having a double point at each of the seven fundamental points. It is of genus 3, and is non-hyperelliptic.* The transformation θ_2 interchanges the parts of every composite cubic passing through the seven fundamental points. A line through two such points and the conic through the remaining five interchange. Each fundamental point goes into a rational cubic having its double point whose image is the point. This cubic passes singly through the remaining six fundamental points. Each such fundamental cubic and its associated point are interchanged by θ_2. Hence, the transformation is the Geiser involutorial transformation.†

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* V. Snyder, "On a Special Algebraic Curve Having a Net of Minimum Adjoint Curves," *Bulletin of the American Mathematical Society*, Vol. XIV (1907), pp. 70–74.

† R. Sturm, "Die Lehre von den Geometrischen Verwandtschaften," Vol. IV, Leipzig (1908), pp. 95–98. The equations of the Geiser transformation are given by V. Snyder. "An Application of the (1, 2) Quaternary Correspondence to the Weddle and Kummer Surfaces," *Transactions of the American Mathematical Society*, Vol. II (1911), pp. 354–366.

Groups of Transformations.

6. Let $T_1, T_2, T_3, \ldots, T_n$ be a set of operations generating the group G of the quartic C_4. Let $Q'_1, Q'_2, \ldots, Q'_m$ be their corresponding space transformations. Let $T'_1, T'_2, T'_3, \ldots, T'_n$ be their corresponding Cremona transformations in the plane $w=0$. Let the symbol A, B represent a group generated by A and B. Let the symbol (r, s) be read "possesses an r to s isomorphism with."

Subgroups of G may be found

$$[T'_1, \theta_2]\,(1, 1)\,[Q'_1, \Gamma]\,(2, 1)\,[T_1],$$
$$[T'_1, T'_j, \theta_2]\,(1, 1)\,[Q'_1, Q'_j, \Gamma]\,(2, 1)\,[T_1, T_j],$$
$$[T'_1, T'_j, T'_k, \theta_2]\,(1, 1)\,[Q'_2, Q'_j, Q'_k, \Gamma]\,(2, 1)\,[T_1, T_j, T_k].$$

Similarly other subgroups may be found.

Determination of G_{48}.

7. The plane quartic curve

$$C_4 \equiv \begin{Bmatrix} x^4+y^4+z^4+k(x^2y^2+x^2z^2+y^2z^2)=0 \\ u=0 \end{Bmatrix} \qquad (13)$$

is invariant under a group $\overline{G}_{24}$ of linear transformations of order 24.* It is generated by the permutation $\overline{G}_6$ on the three letters x, y, z and the harmonic homology defined by changing the sign of any one of the variables.

Bitangents to C_4.

8. Certain lines in the plane $u=0$ are bitangent to C_4. In order to find them it is necessary to identify four special types. The others are found from the collineations of $\overline{G}_6$. We wish to determine, if possible, a relation between k and μ, such that $y=\mu x$ shall be a bitangent; similarly, a relation between k and λ such that $z=\pm\lambda(x\pm y)$ shall be a bitangent.

Eliminate y from $\begin{Bmatrix} y=\pm\mu x \\ u=0 \end{Bmatrix}$ and the equation of C_4 and we obtain

$$\begin{Bmatrix} x^4+k(1+\mu^2)x^2z^2+(1+k\mu^2+\mu^4)z^4=0, \\ u=0. \end{Bmatrix} \qquad (14)$$

The line is a bitangent when the first member of (14) is a square. The condition for this is

$$k^2(1+\mu^2)^2-4k\mu^2-4k(1+\mu^4)=0. \qquad (15)$$

Solve for k and select that value for which the first member of (13) is not a perfect square. We get

$$k=\frac{4\mu^2}{(1+\mu^2)^2}-2. \qquad (16)$$

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* E. Ciani, "Contributo alla teoria del gruppo di 168 collineationi piane," *Annali di Mathematica*, Series 3, Vol. V (1901), pp. 35–55. See p. 43.

Hence $y=\pm\mu x$ must be a bitangent of the C_4, (13) having the value of k defined by (16), where μ may have any finite value.

Similarly, if we eliminate z between (13) and $z=\pm\lambda(x\pm y)$, place the discriminant of the resulting equation equal to zero, and factor out $(k-2)$ we obtain

$$(k+1)\lambda^4-k\lambda^2-1=0. \tag{17}$$

Solving for λ^2 we find $\lambda^2=+1$ or $\lambda^2=-\dfrac{1}{k+1}$. By substituting the second value of k in (16) we get

$$\lambda=\pm\frac{1+\mu^2}{1-\mu^2}. \tag{18}$$

Hence, the line $\begin{Bmatrix} z=\pm\lambda(x\pm y) \\ u=0 \end{Bmatrix}$ must be bitangent to C_4 for the special values of λ.

By operating on the bitangents with $\overline{G}_6$ we obtain the following bitangents:

(1) $y=-\mu z$,	(14) $y=\mu z$,
(2) $z=-\mu x$,	(15) $z=\mu x$,
(3) $x=-\mu y$,	(16) $x=\mu y$,
(4) $x=\lambda(z-y)$,	(17) $x=\lambda(y-z)$,
(5) $y=\lambda(x-z)$,	(18) $y=\lambda(z-x)$,
(6) $z=\lambda(y-x)$,	(19) $z=\lambda(x-y)$,

(7) $x+y+z=0$,

(8) $x=-\lambda(y+z)$,	(20) $x=\lambda(y+z)$,
(9) $y=-\lambda(z+x)$,	(21) $y=\lambda(x+z)$,
(10) $z=-\lambda(x+y)$,	(22) $z=\lambda(x+y)$,
(11) $x=-\mu z$,	(23) $x=\mu z$,
(12) $y=-\mu x$,	(24) $y=\mu x$,
(13) $z=-\mu y$,	(25) $z=\mu y$,

(26) $-x+y+z=0$,
(27) $x-y+z=0$,
(28) $x+y-z=0$,

The Cubic Surface.

9. The next step is to construct the cubic surface having the properties explained in the general case. Select the four bitangents

$$\begin{Bmatrix} x+y+z=0 \\ u\quad=0 \end{Bmatrix},\ \begin{Bmatrix} -x+y+z=0 \\ u\quad=0 \end{Bmatrix},\ \begin{Bmatrix} x-y+z=0 \\ u\quad=0 \end{Bmatrix},\ \begin{Bmatrix} x+y-z=0 \\ u\quad=0 \end{Bmatrix}.$$

The planes passing through $0\equiv(0,0,0,1)$ and these special lines are evidently

$$x+y+z=0,\quad x-y+z=0,$$
$$-x+y+z=0,\quad x+y-z=0.$$

Take $x+y+z=0$ to be the plane tangent to F_3 at 0. The cubic surface has an equation of the form

$$F_3\equiv(x+y+z)u^2+2p(x^2+y^2+z^2)u+(-x+y+z)(x-y+z)(x+y-z)=0. \quad (19)$$

The tangent cone at $(0,0,0,1)$ has the equation

$$(p^2+1)\Sigma x^4+2(p^2-1)\Sigma x^2y^2=0. \quad (20)$$

Comparing the latter with the first one of the two equations in (13) we notice that

$$\frac{2(p^2-1)}{p^2+1}=k=\frac{4\mu^2}{(1+\mu^2)^2}-2. \quad (21)$$

Hence,

$$p^2=\frac{2+k}{2-k} \quad \text{or} \quad p=\frac{\pm\mu}{\sqrt{1+\mu^2+\mu^4}}.$$

Either sign may be used for p. The cubic surface

$$F_3\equiv(x+y+z)u^2+\frac{2\mu(x^2+y^2+z^2)u}{\sqrt{1+\mu^2+\mu^4}}+(-x+y+z)(x-y+z)(x+y-z)=0 \quad (22)$$

is selected for our discussion. Rearranging (19) we find

$$(x+y+z)u+\frac{2\mu(x^2+y^2+z^2)}{\sqrt{1+\mu^2+\mu^4}}=\pm(p^2+1)\sqrt{\Sigma x^4+\frac{2p^2-1}{p^2+1}\Sigma x^2y^2} \quad (23)$$

which by means of (21) can be reduced to the form

$$\{(x+y+z)v+\mu(x^2+y^2+z^2)\}^2=(1+\mu^2)\{\Sigma x^4+k\Sigma x^2y^2\}, \quad (24)$$

where $v=u\sqrt{1+\mu^2+\mu^4}$ and $k=\dfrac{4\mu^2}{(1+\mu^2)^2}-2$.

Equation (24) defines the composite surface F_3 and $(x+y+z)=0$.

The Lines on the Cubic Surface.

10. Each plane determined by 0 and some one of the twenty-seven bitangents to the quartic C_4 is a bitangent plane to F_3. The plane $x+y+z=0$, containing a bitangent to C_4, is tangent to F_3 at 0. It meets F_3 in a non-composite cubic curve.

We find each of the 27 lines upon F_3, together with their residual conics in the plane through 0 by solving their equations simultaneously with the equation of the composite surface (24).

The latter equation is reduced to the form

$$(x+y+z)v+\mu(x^2+y^2+z^2)=\pm(1+\mu^2)\cdot\phi_2, \tag{25}$$

where ϕ_2 corresponds to ϕ_2 in equation (6). One sign in the right member corresponds to a line on F_3 when one of the twenty-seven special planes is taken simultaneously with it. The opposite sign corresponds to a conic section lying in the bitangent plane used.

Equation (7) needs the same consideration as was given to the plane $f_1(x,y,z)=0$ in Section 3. We may show by the same reasoning as was employed there that the positive sign in the right member of (25) corresponds to the cubic curve

$$\left.\begin{aligned} 2\mu(x^2+y^2+z^2)u+(-x+y+z)(x-y+z)(x+y-z)&=0,\\ x+y+z&=0.\end{aligned}\right\} \tag{26}$$

The following list of equations represents the lines and the residual conic sections unless otherwise stated. The equations are numbered to correspond with the bitangents given previously. The first two equations in each trio fix a line, the first and third fix the residual conic sections in the plane connecting the line with 0. The third equation is added for a purpose which will be explained later.

(1)
$$\begin{gathered} y+\mu z=0\\ v+(1+\mu+\mu^2)(x-\overline{1-\mu}\cdot z)=0\\ (x+y+z)v+\mu(x^2+y^2+z^2)\\ =+\{(1+\mu^2)x^2-(1+\mu^4)z^2\}\end{gathered}$$

(4)
$$\begin{gathered} x=\lambda(z-y)\\ v+\mu\{(1-\lambda)z+(1+\lambda)y\}=0\\ (x+y+z)v+\mu(x^2+y^2+z^2)\\ =-2\mu\cdot\lambda^2\left\{z^2+y^2-\frac{2\lambda^2+1}{\lambda^2}\cdot yz\right\}\end{gathered}$$

(7)
$$\begin{gathered} x+y+z=0 \text{ corresponds to } 0\equiv(0,0,0,1)\\ (x+y+z)v+\mu(x^2+y^2+z^2)\\ =+2\mu(x^2+z^2+xz) \text{ corresponds to the cubic curve}\\ \text{in the tangent plane.}\end{gathered}$$

(8)
$$\begin{gathered} x=-\lambda(y+z)\\ (1-\lambda)v+\mu(1+3\mu^2)(y+z)=0\\ (x+y+z)v+\mu(x^2+y^2+z^2)\\ =+2\mu\cdot\lambda^2\left\{y^2+z^2+\frac{2\lambda^2+1}{\lambda^2}yz\right\}\end{gathered}$$

(26)
$$-x+y+z=0$$
$$v=0$$
$$(x+y+z)v+\mu(x^2+y^2+z^2)$$
$$=-2\mu\{y^2+z^2+yz\}$$

(10)
$$z=-\lambda(x+y)$$
$$(1-\lambda)v+\mu(1+3\mu^2)(x+y)=0$$
$$(x+y+z)v+\mu(x^2+y^2+z^2)$$
$$=+2\mu\cdot\lambda^2\cdot\left\{x^2+y^2+\frac{2\lambda^2+1}{\lambda^2}xy\right\}$$

(11)
$$x=-\mu z$$
$$v+(1+\mu+\mu^2)(y-\overline{1-\mu}\,z)=0$$
$$(x+y+z)v+\mu(x^2+y^2+z^2)$$
$$=+\{(1+\mu^2)y^2-(1+\mu^4)z^2\}$$

(12)
$$y=-\mu x$$
$$v+(1+\mu+\mu^2)(z-\overline{1-\mu}\,x)=0$$
$$(x+y+z)v+\mu(x^2+y^2+z^2)$$
$$=+\{(1+\mu^2)z^2-(1+\mu^4)x^2\}.$$

Apply $\begin{pmatrix} x\,y\,z \\ y\,z\,x \end{pmatrix}$ to (1), (2), (4), (5), (8), (9) to get (2), (3), (5), (6), (9), (10). Write $-\mu$ for μ, $-v$ for v in (1), (2), (3) to get (14), (15), (16) and $-\lambda$ for λ in (4), (5), (6), (8), (9), (10) to get (17), (18), (19), (20), (21), (22). Interchange x and y in (3), (16), (26) to get (12), (24), (27); y and z in (1), (14), (27) to get (13), (25), (28); z and x in (2), (15) to get (11), (23).

Arrangement of the 27 Lines on F_3.

11. A well-known property of a non-singular cubic surface F_3 is that each line on F_3 is contained in a set of five planes each of which contains two other lines on F_3. By testing the equations in Section 10, we find that the line (26) forms a triangle with each pair of lines [(14), (25)], [(20), (8)], [(17), (4)], [(1), (13)], [(28), (27)] and, similarly, (28) with [(16), (24)], [(22), (10)], [(19), (6)], [(3), (12)], [(26), (27)], and (27) with [(15), (23)], [(21), (9)], [(18), (5)], [(2), (11)], [(28), (26)].

If we call $(26)\equiv C_{14}$* and $(16)\equiv C_{12}$, then (27) is a c line which we shall designate by c_{25}; (24) by c_{45} and (28) by c_{36}.

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* Schläfli's notation has been adopted. F. Schläfli, "An Attempt to Determine the Twenty-seven Lines upon a Surface of the Third Order and to Divide Such Surfaces into Species in Reference to the Reality of the Lines upon the Surface," *Quarterly Journal*, Vol. II (1858), pp. 55–65 and 110–120; see p. 115. K. Doehlemann, "Geometrische Transformationen," Teil II (1907), pp. 303–305.

The only four lines skew to (26) and (24) but intersecting (16) and (27) are (21), (23), (2) and (18), which must consist of two c lines, one a and one b line. Call them c_{34}, c_{46}, a_2, b_2, respectively. The only four lines skew to (16) and (27), but intersecting (26) and (24) are (14), (20), (4) and (13). Of these (14) is skew (28) and, therefore, must be c_{23}. Similarly, (20), (4) and (13) must be c_{26}, a_4, b_4, respectively. We see now that $(25)\equiv c_{56}$, $(8)\equiv c_{35}$, $(17)\equiv b_1$, $(1)\equiv a_1$, $(15)\equiv c_{13}$, $(9)\equiv c_{16}$, $(5)\equiv a_5$, $(11)\equiv b_5$, leaving unlettered the lines [(22), (10)], [(19), (6)], [(3), (12)].

Consider the double six

$$\begin{matrix} c_{14}, & c_{45}, & c_{34}, & c_{24}, & a_6, & b_6, \\ c_{16}, & c_{56}, & c_{36}, & c_{26}, & a_4, & b_4, \end{matrix}$$

(25) meets (10), (6) and (12) which must be c_{24}, a_6, b_6 in the same order, but (20) is skew to (10), (6) to (4), (12) to (13); hence, the order is correct. This completely determines the lettering of the lines which is as follows:

$(1)\equiv a_1$	$(8)\equiv c_{35}$	$(15)\equiv c_{13}$	$(22)\equiv c_{15}$
$(2)\equiv a_2$	$(9)\equiv c_{16}$	$(16)\equiv c_{12}$	$(23)\equiv c_{46}$
$(3)\equiv a_3$	$(10)\equiv c_{24}$	$(17)\equiv b_1$	$(24)\equiv c_{45}$
$(4)\equiv a_4$	$(11)\equiv b_5$	$(18)\equiv b_2$	$(25)\equiv c_{56}$
$(5)\equiv a_5$	$(12)\equiv b_6$	$(19)\equiv b_3$	$(26)\equiv c_{14}$
$(6)\equiv a_6$	$(13)\equiv b_4$	$(20)\equiv c_{26}$	$(27)\equiv c_{25}$
	$(14)\equiv c_{23}$	$(21)\equiv c_{34}$	$(28)\equiv c_{36}$

Instead of choosing (26) as a c line we might have taken it for an a_1 or b_1 line in which cases (16) would be b_1 or a_1, respectively. Hence, we may name the lines on F_3 in a variety of ways.

Identification of the Generators of G_{45}.

12. Select a transformation $T_3\equiv\begin{pmatrix} x, & y, & z \\ y', & z', & x' \end{pmatrix}$ which leaves C_4 invariant. (The subscript in T_3 denotes the period.) Choose the transformation Q_3' which is defined by the equations

$$\begin{cases} x = y' \\ y = z' \\ z = x' \\ v = \dfrac{-\mu(x'^2+y'^2+z'^2)+(x'+y'+z')v'+\mu(x^2+y^2+z^2)}{x+y+z} \end{cases}$$

which in our case reduces to the linear space transformation

$$Q_3' = \begin{pmatrix} x, & y, & z, & v \\ y', & z', & x', & v' \end{pmatrix}.$$

It is one of the two space transformations which correspond to T_3. The transformation Q_3' permutes the twenty-eight planes through 0 among themselves. It leaves invariant the left member of each second degree equation listed in Section 10. It may or may not change the right member. The same is true of those second degree equations from which those planes are found, which with the bitangent planes fix the lines on F_3. The transformation Q_3' is operated as follows: Take for example the line a_1. The transformation $T_3 \equiv \begin{pmatrix} x, & y, & z \\ y', & z', & x' \end{pmatrix}$ transforms the bitangent $\begin{Bmatrix} y = -\mu z \\ u = 0 \end{Bmatrix}$ into the bitangent $\begin{Bmatrix} z = -\mu x \\ u = 0 \end{Bmatrix}$; hence, Q_3' changes the plane $y = -\mu z$ into the plane $z = -\mu x$. The left member of the third equation under (1) remains invariant under Q_3'. The right changes but it is changed only by the equations $\begin{Bmatrix} x = y' \\ y = z' \\ z = x' \end{Bmatrix}$ which define T_3. Therefore, change the sign of the right member of the quadratic equation given; as,

$$-\{(1+\mu^2)x^2-(1+\mu^4)z^2\}.$$

This corresponds to the line a_1 itself. Operate on this expression by T_3 and obtain

$$-\{(1+\mu^2)y^2-(1+\mu^4)x^2\};$$

the latter corresponds to the line a_2. Hence, we conclude that a_1 goes into a_2.

In applying this process we find that Q_3' transforms $a_1, a_2, a_3, a_4, a_5, a_6$ and the point 0 into $a_2, a_3, a_1, a_5, a_6, a_4$ and the point 0, respectively.

The images of $a_1, a_2, a_3, a_4, a_5, a_6$ and the point 0 in the plane $w=0$ are the fundamental points 1, 2, 3, 4, 5, 6, 7.* The images of the b_i lines are the conics $[i7]$. The image of the residual of b_i is the line $(i, 7)$. The image of the line c_{ik} is the line (i, k). The image of its residual is the conic $[ik]$. The image of the residuals of 0, a_i are the cubics $[7]$ $[i]$.

Hence, the corresponding Cremona transformation is

$$T_3' \equiv \begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ 2, & 3, & 1, & 5, & 6, & 4, & 7 \end{pmatrix}.$$

This notation means that the image of 1 is 2, and so on for the other fundamental points. Since a transformation is fixed when the images of its fundamental points are known, we have our required transformation which is linear and of period 3.

If we had taken the transformation Q_3'' which also corresponds to T_3, we would have been led to a new transformation T_3''. This can also be found by

* The symbol (ij) denotes the line ij, $[ij]$ the conic through the remaining five fundamental points, and $[i]$ the cubic with a double point at i which passes through the other six fundamental points.

operating on T_3' by θ_2, the Geiser transformation. It transforms the point 1 into the residual of its image in the transformation T_3'. Similarly for the other fundamental points. Therefore,

$$T_3'' \equiv \begin{Bmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [2], & [3], & [1], & [5], & [6], & [4], & [7] \end{Bmatrix}$$

which is of order 8.

The transformation $T_2 \equiv \begin{Bmatrix} x, y, z \\ -x', y', z' \end{Bmatrix}$ leads to the transformation

$$T_2' \equiv \begin{Bmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ 1, & [13], & [12], & [17], & [16], & [15], & [14] \end{Bmatrix}.$$

The transformation $T_2'' \equiv T_2' \cdot \theta_2$ is

$$T_2'' \equiv \begin{Bmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [1], & (13), & (12), & (17), & (16), & (15), & (14) \end{Bmatrix}$$

which is a Jonquières transformation* of order 4.

Use the same fundamental process on $\bar{T}_2 \equiv \begin{pmatrix} x, y, z \\ y, x, z \end{pmatrix}$. We are led to

$$\bar{T}_2' \equiv \begin{Bmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [57], & [47], & [67], & [27], & [17], & [37], & [7] \end{Bmatrix}$$

which is of order 5 and

$$\bar{T}_2'' \equiv \begin{Bmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ (57), & (47), & (67), & (27), & (17), & (37), & [7] \end{Bmatrix}$$

which is a Jonquières transformation of order 4.

G_{48} *with Some of its Subgroups.*

13. Using the notation of Section 6 we can arrive at different groups:

$G_4 \equiv [T_2', \theta_2]\,(1,1)\,[Q_2', \Gamma]\,(2,1)\,[T_2] \equiv \bar{G}_2$ on C_4,
$G_6 \equiv [T_3', \theta_2]\,(1,1)\,[Q_3', \Gamma]\,(2,1)\,[T_3] \equiv \bar{G}_3$ on C_4,
$G_4 \equiv [\bar{T}_2', \theta_2]\,(1,1)\,[\bar{Q}_2', \Gamma]\,(2,1)\,[T_2'] \equiv \bar{G}_2$ on C_4,
$G_{12} \equiv [T_3', T_2', \theta_2]\,(1,1)\,[Q_3', Q_2', \Gamma]\,(2,1)\,[T_3, T_2] \equiv \bar{G}_6$ on C_4,
$G_{16} \equiv [\bar{T}_2', T_2', \theta_2]\,(1,1)\,[\bar{Q}_2', Q_2', \Gamma]\,(2,1)\,[T_2, T_2] \equiv \bar{G}$ on$_8$ C_4,
$G_{24} \equiv [T_3', T_2', \theta_2]\,(1,1)\,[Q_3', Q_2', \Gamma]\,(2,1)\,[T_3, T_2] \equiv \bar{G}_{12}$ on C_4,
$G_{48} \equiv [T_3', T_2', \bar{T}_2', \theta_2]\,(1,1)\,[Q_3', Q_2', \bar{Q}_2', \Gamma]\,(2,1)\,[T_3, T_2, T_2'] \equiv \bar{G}_{24}$.

On account of the isomorphism existing between the groups G_{2n} and the groups $\bar{G}_n$ many other groups may be found such that groups G_m are simply isomorphic to $\bar{G}_m$.

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* K. Doehlemann, "Geometrische Transformationen," Teil II (1907), p. 150.

14. Since the value of μ in our previous discussion may have any finite value, we may expect to find groups of higher order than 48 by restricting the value of μ.

Determination of G_{192}.

15. Let $k=0$, hence from the equation $\frac{4\mu^2}{(1+\mu^2)^2}-2=0$, we deduce

$$\mu^4+1=0. \tag{27}$$

Therefore, the quartic curve (13) has an equation of the form

$$\left\{\begin{matrix} x^4+y^4+z^4=0 \\ u \quad =0 \end{matrix}\right\}, \tag{28}$$

which has been studied by Dyck.* This curve is invariant under the group $\bar{G}_{96}$ of order 96. The octahedral group $\bar{G}_{24}$ and the transformation $T_4 \equiv \begin{pmatrix} x, & y, & z \\ ix', & y', & z' \end{pmatrix}$ generate it. Hence, in order to determine the corresponding group G_{192} all that remains is to determine a transformation in the plane $w=0$ corresponding to T_4.

Repeat the argument precisely as given in the previous case understanding that μ is subject to the condition

$$\mu^4+1=0$$

in all of the previous equations.

We are led to the transformations T_4' of order 6,

$$\left\{\begin{matrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [1], & [46], & [67], & [36], & (15), & [5], & [26] \end{matrix}\right\}$$

$$\text{and } T_4'' \equiv T_4' \cdot \theta_2 \equiv \left\{\begin{matrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ (1), & (46), & (67), & (36), & [15], & (5), & (26) \end{matrix}\right\}$$

is a Jonquière transformation of order 3.

T_4' and T_4'' correspond to $T_4 \equiv \left\{\begin{matrix} x, & y, & z \\ ix', & y', & z' \end{matrix}\right\}$.

Hence, the group $G_{192} \equiv [G_{24}, T_4', \theta_2]\,(2,1)$ the group $\bar{G}_{96} \equiv [\bar{G}_{24}, T_4]$.

Determination of G_{336}.

16. Consider the case where μ satisfies the equation

$$\mu^2-\mu+2=0. \tag{29}$$

The value of k subject to this restriction is -3μ. Hence, the quartic C_4 reduces to the form

$$C_4 \equiv \left\{\begin{matrix} x^4+y^4+z^4-3\mu(x^2y^2+x^2z^2+y^2z^2)=0 \\ u \qquad\qquad =0 \end{matrix}\right\}. \tag{30}$$

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* W. Dyck, "Notiz über eine reguläre Riemannische Fläche vom Geschlechte drei und die zugehörige Normalcurve vierter Ordnung," *Mathematische Annalen*, Vol. XVIII (1881), pp. 510–516; see p. 512.

Ciani* states that this quartic, subject to the condition (29), can be reduced by a suitable linear transformation to the form

$$\left\{\begin{matrix} x^3 z + y^3 x + z^3 y = 0 \\ u \quad = 0 \end{matrix}\right\}. \tag{31}$$

Klein† has shown that the latter admits a group of collineations $\overline{G}_{168}$ which is generated by an octahedral group $\overline{G}_{24}$ and a certain cyclic group $\overline{G}_7$.

Hence, the quartic in the form given by Ciani is invariant under a group $\overline{G}_{168}$ which is generated by our octahedral group $\overline{G}_{24}$ and some $\overline{T}_7$. Such a transformation

$$T_7 \equiv \left\{\begin{matrix} x = -[2x' - \mu(y' - z')] \\ y = 2x' + \mu(y' - z') \\ z = \mu^2(y' + z') \end{matrix}\right\}$$

is given by Sharpe. ‡

Hence, we repeat the argument which is given for G_{48} with the restriction that μ satisfies the equation

$$\mu^2 - \mu + 2 = 0$$

in order to determine a transformation T_7' corresponding to T_7. Since some care is necessary in determining the images of $a_1, a_2, a_3, a_4, a_5, a_6$ and the point 0, the details in finding the image of a_1 are given.

First operate on the quartic

$$\left\{\begin{matrix} x^4 + y^4 + z^4 - 3\mu(x^2 y^2 + x^2 z + y^2 z^2) = 0 \\ u \quad = 0 \end{matrix}\right\}$$

with the transformation T_7. Select the special term containing x^4 which we find to be

$$16(2+k)x^4.$$

Since
$$k = \frac{4\mu^2}{(1+\mu^2)^2} - 2,$$

therefore,
$$16(2+k)x^4 = \frac{(16)(4)\mu^2}{(1+\mu^2)^2} \cdot x^4.$$

Hence, in extracting the root we find the factor

$$\sqrt{16(2+k)} = \pm \frac{8\mu}{1+\mu^2},$$

* E. Ciani, "I Varii Tipi Possibili di Quartiche Piane più Volte Omologico Armoniche," *Rendiconti del circolo Matematico di Palermo*, Vol. XIII (1899), pp. 347–373; see p. 365.

† F. Klein, "Ueber die Transformation siebenter Ordnung der Elliptischen Functionen, *Mathematische Annalen*, Vol. XIV (1879), pp. 428–471; see p. 446.

‡ F. R. Sharpe, "Conics through Inflections of Self-Projective Quartics," American Journal of Mathematics, Vol. XXXVII (1915), pp. 66–72; see p. 71.

but since $\mu^2-\mu+2=0$,

$$\sqrt{16(2+k)}=-\frac{8\mu}{1+\mu^2}=4\mu^2.$$

Hence, in making certain comparisons the factor $4\mu^2$ must be taken into account.

We now operate with the Q_7' corresponding to T_7 as follows: The line a_1 is determined by

$$y=-\mu z$$
$$(x+y+z)v+\mu(x^2+y^2+z^2)=-\{(1+\mu^2)x^2-(1+\mu^4)z^2\},$$

or

$$y=-\mu z$$
$$(x+y+z)v+\mu(x^2+y^2+z^2)=-(\mu-1)\{x^2+3z^2\}.$$

Operate on $y=-\mu z$ *by* T_7 and we get the equation

$$z=\frac{\mu^2}{4}(y-x). \tag{32}$$

Hence, a_1 goes into the line a_6 or its residual. To decide which, operate on

$$-(\mu-1)\{x^2+3z^2\} \tag{33}$$

by
$$x=-[2x'-\mu(y'-z')],\quad z=\frac{\mu^2}{4}(y'+z'), \tag{34}$$

where $z'=\frac{\mu^2}{4}(y''-x'')$ as in equation (32).

The coefficient of x''^2 reduces to $-\frac{1}{2}\mu^7$, but a_6 is defined by

$$z=\frac{\mu^2}{4}(y-x),(x+y+z)v+\mu(x^2+y^2+z^2)=\frac{+2\mu^6}{16}\left\{y^2+x^2-\frac{2\lambda^2+1}{\lambda^2}\cdot xy\right\}. \tag{35}$$

Compare the x''^2 term with the corresponding term in (35) remembering the former contains an extra factor $4\mu^2$, due to operating with T_7. Hence, we conclude that a_1 goes into the residual of a_6. Similarly, we show that a_3 and 0 go into the lines c_{36}, c_{56} while a_2, a_4, a_5, a_6 go into the residuals of c_{12}, b_6, c_{24}, c_{14}, respectively. Hence, in the w plane we get the transformations T_7' of order 5,

$$\begin{Bmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [6], & [12], & (36), & (67), & [24], & [14], & (56) \end{Bmatrix}$$

and
$$T_7'\cdot\theta_2\equiv T_7''\equiv\begin{Bmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [6], & (12), & [36], & [67], & (24), & (14), & [56] \end{Bmatrix}.$$

Hence, the group $C_{336}\equiv[G_{24}, T_7', \theta_2](2,1)$, the group $\overline{G}_{168}\equiv[\overline{G}_{24}, T_7]$.

Cornell University, *July*, 1915.

*On the Determination of a Certain Class of Surfaces.**

BY ARCHER EVERETT YOUNG.

Introduction.

The second fundamental form for a surface S referred to any set of conjugate lines can be written †

$$D d\bar{u}^2 + D'' d\bar{v}^2.$$

If the surface in question has positive total curvature there exists an infinity of conjugate systems whose choice as lines of reference, the parameters being properly chosen, reduce this form to

$$D(d\bar{u}^2 + d\bar{v}^2).$$

These particular conjugate systems have been named ‡ isotherm-conjugate. If, on the other hand, the surface has negative total curvature, there exists an infinity of conjugate systems whose choice as lines of reference permit the second form to be written

$$D(d\bar{u}^2 - d\bar{v}^2).$$

We shall, for convenience, refer to such as associate isotherm-conjugate lines. Many well-known surfaces of positive total curvature have *lines of curvature* which belong to the class defined as isotherm-conjugate, while others of negative total curvature have lines of curvature which are associate isotherm-conjugate.

We have discussed in this paper the *problem of determining all surfaces having lines of curvature which are either isotherm-conjugate or associate isotherm-conjugate.*

We show in Article I that on the surfaces having negative total curvature, the asymptotic lines form a network of infinitesimal rhombi, while on those of positive the characteristic lines § form a similar network. We may say,

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* Presented to the American Mathematical Society, December 31, 1915.

† Bianchi, "Lezioni di Geometria Differenziale," p. 113.

‡ Bianchi, *loc. cit.*, p. 168.

§ A name given by Pucci to that system of lines of positive total curvature, for which when it is parametric, $\frac{D}{E} = \frac{D''}{G}$, $\frac{D}{e} = \frac{D''}{g}$, $D' = 0$ ("Dell' angolo caratteristico e delle linee caratterische di una superficie," *Rom. Acc. L. Rend.*, IV (1889), pp. 501–507).

therefore, that our *problem is that of determining all surfaces on which either the asymptotic or the characteristic lines form a network of infinitesimal rhombi.*

The problem is treated in Article II from the standpoint of the spherical representation of the asymptotic lines for surfaces having negative total curvature, and characteristic lines for surfaces having positive total curvature. After the derivation of the system of equations, on the solution of which the general problem depends, certain particular solutions are considered in Article III. In the closing Article IV, we have reduced the problem to the solution of the fundamental equations used in the Bonnet form, with the addition of another equation in the same functions. We are enabled thus to point out a general method for determining all surfaces of the problem which are solutions of the general "Problem of the Spherical Representation".

I. *The Characteristic Geometric Properties of the Surfaces.*

Let $d\bar{s}^2 = Edu^2 + 2Fdudv + Gd\bar{v}^2$ be the expression for the linear element of a surface S of negative total curvature referred to asymptotic lines. The equation of the characteristic lines, which in general is*

$$[D(GD-ED'')-2D'(FD-ED')]d\bar{u}^2+[2D'(GD+ED'')-4FDD'']dudv \\ +[2D'(GD'-FD'')-D''(GD-ED'')]d\bar{v}^2=0,$$

reduces to

$$Ed\bar{u}^2+Gd\bar{v}^2=0,$$

since the lines of reference are asymptotic. They are, therefore, imaginary on a surface of negative total curvature.

If the asymptotic lines on the surface chosen, form a network of infinitesimal rhombi, the expression above by a proper choice of parameters reduces to

$$d\bar{s}^2=\lambda(d\bar{u}^2+2\cos\theta\, dudv+d\bar{v}^2), \tag{I}$$

where θ is the angle between the asymptotic lines.

The equations of the characteristic lines and of the lines of curvature on the surface corresponding to (I) are, respectively, $d\bar{u}^2+d\bar{v}^2=0$, and $d\bar{u}^2-d\bar{v}^2=0$.

Referring the surface to lines of curvature by writing $du+dv=du_1$, $du-dv_1=dv_1$, (I) becomes† $d\bar{s}^2=\frac{\lambda}{2}\Big((1+\cos\theta)d\bar{u}_1^2+(1-\cos\theta)d\bar{v}_1^2\Big)$, and the equation of the asymptotic lines is now $d\bar{u}_1^2-d\bar{v}_1^2=0$. Hence the theorem:

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* Eisenhart, *Transactions of A. M. S.*, Vol. V, pp. 421–437.

† Knoblauch, "Einleitung in die Allgemeine Theorie der Krummen Flächen," p. 12.

If the asymptotic lines on a surface S of negative total curvature form on the surface a network of infinitesimal rhombi, the lines of curvature must be associate isotherm-conjugate. The converse is easily proven.

From the form of the expression for the linear element just above, it appears that the corresponding surface will be *isothermic*; that is, *have isothermal lines of curvature*, when and only when

$$\tan\frac{\theta}{2}=\frac{f(u_1)}{\phi(v_1)},$$

where, of course, θ is the angle between the asymptotic lines, and hence the theorem:

A surface of negative total curvature which has associate isotherm-conjugate lines of curvature will be isothermic when and only when the tangent of one-half the angle between the asymptotic lines is equal to a function of one parameter divided by a function of the other, the lines of reference being lines of curvature.

Turning now to the consideration of surfaces having positive total curvature, let $d\bar{s}_1^2=E_1d\bar{u}^2+2F_1dvdu+G_1d\bar{v}^2$ be the expression for the linear element of a surface referred to characteristic lines.

Since $\dfrac{D_1}{E_1}=\dfrac{D_1''}{G_1}$, $D_1'=0$, the equation of the asymptotic lines may be written thus:

$$E_1\,du^2+G_1\,d\bar{v}^2=0.$$

If the characteristic lines form a network of rhombi, then, by a proper choice of parameters, the expression for the linear element becomes

$$d\bar{s}_1^2=\lambda_1(du^2+2\cos\theta_1\,dudv+d\bar{v}^2), \qquad \text{(I')}$$

where θ_1 is the angle between the characteristic lines. The equations of the asymptotic lines and of the lines of curvature on a surface which corresponds to (I') are, respectively,

$$d\bar{u}^2+d\bar{v}^2=0,\ \text{and}\ d\bar{u}^2-d\bar{v}^2=0. \qquad \text{(II)}$$

Taking the lines of curvature for lines of reference, and following the reasoning of the preceding case, we easily prove the following theorems:

If a surface of positive total curvature has characteristic lines which divide the surface into a network of infinitesimal rhombi, its lines of curvature are isotherm-conjugate; and, conversely.

The necessary and sufficient condition that a surface of positive total curvature, which has isotherm-conjugate lines of curvature, shall be isothermic, is that the tangent of one-half the angle between the characteristic lines shall be equal to a function of one parameter over a function of the other, the lines of reference being lines of curvature.

II. *Discussion of the General Problem.*

If (I) is the first form for a surface S of negative total curvature, referred to asymptotic lines, the third form may be written thus:*

$$d\bar{\sigma}^2=\mu(d\bar{u}^2-2\cos\theta\, du dv+d\bar{v}^2), \tag{III}$$

where $\mu=\dfrac{\lambda}{-K}$, K being the total curvature.

Likewise, if (I′) is the first form for a surface S_1 of positive total curvature, referred to characteristic lines, the third form† is

$$d\bar{\sigma}_1^2=\mu_1(d\bar{u}^2-2\cos\theta_1\, du\, dv+d\bar{v}^2), \tag{III′}$$

where $\mu_1=\lambda_1/K_1$, K_1 as before being the total curvature.

If (III) is the third form for a surface S, referred to asymptotic lines, the functions involved must satisfy the following equation:‡

$$\frac{\partial}{\partial u}\left[\frac{\dfrac{\partial\log\mu}{\partial v}+\cos\theta\dfrac{\partial\log\mu}{\partial u}}{\sin^2\theta}\right]=\frac{\partial}{\partial v}\left[\frac{\partial\log\mu}{\partial u}+\cos\theta\frac{\partial\log\mu}{\partial v}\right]; \tag{1}$$

and, similarly, if (III′) is the third form for a surface S_1 referred to characteristic lines, §

$$\frac{\partial}{\partial u}\left[\frac{\dfrac{\partial}{\partial u}(\mu_1\cos\theta_1)+\cos\theta_1\dfrac{\partial(\mu\cos\theta_1)}{\partial v}}{\mu_1\sin^2\theta_1}\right]=\frac{\partial}{\partial v}\left[\frac{\dfrac{\partial}{\partial v}(\mu_1\cos\theta_1)+\cos\theta_1\dfrac{\partial(\mu_1\cos\theta_1)}{\partial u}}{\mu_1\sin^2\theta_1}\right]. \tag{2}$$

For the purpose of comparison we write equations (1) and (2) in the following forms, respectively:

$$\begin{aligned}\frac{\partial^2\log\mu}{\partial u^2}-\frac{\partial^2\log\mu}{\partial v^2}&+\frac{\partial\log\mu}{\partial u}\left(\frac{\partial}{\partial u}\log\left(\frac{\cos\theta}{\sin^2\theta}\right)+\frac{\partial}{\partial v}\log\tan^2\frac{\theta}{2}\right)\\&-\frac{\partial\log\mu}{\partial v}\left(\frac{\partial}{\partial v}\log\left(\frac{\cos\theta}{\sin^2\theta}\right)+\frac{\partial}{\partial u}\log\tan^2\frac{\theta}{2}\right)=0;\end{aligned} \tag{1′}$$

$$\begin{aligned}\frac{\partial^2\log\mu_1}{\partial u^2}-\frac{\partial\log\mu_1}{\partial v^2}&+\frac{\partial\log\mu_1}{\partial u}\left(\frac{\partial}{\partial u}\log\frac{\cos\theta_1}{\sin^2\theta_1}+\frac{\partial}{\partial v}\log\tan^2\frac{\theta_1}{2}\right)\\&-\frac{\partial\log\mu_1}{\partial v}\left(\frac{\partial}{\partial v}\log\frac{\cos\theta_1}{\sin^2\theta_1}+\frac{\partial}{\partial u}\log\tan^2\frac{\theta_1}{2}\right)\\&+\frac{\partial^2}{\partial u^2}\log\tan\frac{\theta_1}{2}-\frac{\partial^2}{\partial v^2}\log\tan\frac{\theta_1}{2}=0.\end{aligned} \tag{2′}$$

* Bianchi, *loc. cit.*, p. 156. † Bianchi, *loc. cit.*, p. 168. ‡ *Ibid.*, p. 156. § *Ibid.*, p. 169.

The Gauss equation which the functions appearing in (1′) and (2′), respectively, must satisfy, may be written thus:

$$\csc^2\frac{\theta}{2}\left(\frac{\partial^2 \log\mu}{\partial u^2}+\frac{\partial^2 \log\mu}{\partial v^2}\right)-2\left(1-\cot^2\frac{\theta}{2}\right)\frac{\partial^2 \log\mu}{\partial u\partial v}-\frac{\partial}{\partial u}\log\tan^2\frac{\theta}{2}\frac{\partial \log\mu}{\partial u}$$
$$-\frac{\partial}{\partial v}\log\tan^2\frac{\theta}{2}\frac{\partial \log\mu}{\partial v}+8\cos^2\frac{\theta}{2}\mu-4\cot\frac{\theta}{2}\frac{\partial^2\theta}{\partial u\partial v}=0, \qquad (3)$$

where μ and θ are to be replaced, respectively, by μ_1 and θ_1 when used with (2′).

The general problem, then, is reduced to the solutions of the pairs of equations (1′), (3) *and* (2′), (3); *the first leading to the surfaces of negative total curvature which are solutions of the problem, and the second to those of positive.*

As the general solution of these systems of equations can not be obtained, the general solution of the problem, from this standpoint, at least, is denied us. Particular solutions are, however, easily obtained, and the method of procedure is as follows:

Having obtained a solution of the pair of equations (1′), (3) or (2′), (3), we have the third form for the corresponding surfaces referred, in the first case to asymptotic lines, and in the second to characteristic lines. The first forms, or the expressions for the linear elements on the surfaces, are then obtained by quadrature from one or the other of the following systems of equations: *

$$\left\{\begin{aligned}\frac{\partial\log\rho}{\partial u}&=\frac{-\cos\theta\dfrac{\partial\log\mu}{\partial v}-\dfrac{\partial\log\mu}{\partial u}}{\sin^2\theta}\\ \frac{\partial\log\rho}{\partial v}&=\frac{-\cos\theta\dfrac{\partial\log\mu}{\partial u}-\dfrac{\partial\log\mu}{\partial v}}{\sin^2\theta}\end{aligned}\right\}, \qquad (4)$$

where $K=-1/\rho^2$;

$$\left\{\begin{aligned}\frac{\partial\log\rho_1}{\partial u}&=\frac{\cos\theta_1\dfrac{\partial}{\partial u}(\mu_1\cos\theta_1)+\dfrac{\partial}{\partial v}(\mu_1\cos\theta_1)}{\mu_1\sin^2\theta_1}\\ \frac{\partial\log\rho_1}{\partial v}&=\frac{\cos\theta_1\dfrac{\partial}{\partial v}(\mu_1\cos\theta_4)+\dfrac{\partial}{\partial u}(\mu_1\cos\theta_1)}{\mu_1\sin^2\theta_1}\end{aligned}\right\} \qquad (4')$$

where $K_1=1/\rho^2$.

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* Bianchi, *loc. cit.*, p. 155 and p. 168.

The Cartesian coordinates may then be obtained, following the general method, by the integration of a Ricatti equation or by some special device, depending upon the functions involved.

Before closing this article we shall call attention to some particular solutions which are suggested by the form of the equations (1), (2) and (3).

CASE 1. If in particular the function μ reduces to a constant, the corresponding surfaces are pseudo-spherical. All pseudo-spherical surfaces are included here. Likewise, if the function μ_1 reduces to a constant, the integration of the equations (2′) and (3) leads to the surfaces of constant positive total curvature.

CASE 2. If $\cos\theta=f(u+v)$, equation (1′) may be written in the form

$$\frac{\partial^2 \log\mu}{\partial u^2}-\frac{\partial^2 \log\mu}{\partial v^2}-\phi(u+v)\left(\frac{\partial \log\mu}{\partial u}-\frac{\partial \log\mu}{\partial v}\right)=0, \tag{5}$$

where ϕ being a function of θ must be a function of $u+v$ as indicated. The general solution of (5) is $\log\mu=F(u+v)\psi(u-v)+\chi(u+v)$, where ψ and χ are arbitrary functions, and $\phi=2F'/F$.

Substituting this value for $\log\mu$ in the Gauss equation (3), changing parameters by writing $u+v=u_1$, $u-v=v_1$, differentiating, and separating functions of u_1 from functions of v_1, we show that the function ψ must reduce to a constant. It follows easily that the corresponding surfaces are surfaces of revolution. All surfaces of revolution having negative total curvature are included here. In a similar way, we obtain the surfaces of revolution having positive total curvature, from the solution of (2′) and (3), assuming that the $\cos\theta_1$ is a function of the sum of the two variables.

If, in particular, θ is a constant, the integration of (1′), (3) and (4) gives as the linear element of the corresponding surfaces the following:

$$d\bar{s}^2=\left[\frac{(1-be^{c(u+v)})^2}{e^{c(u+v)}}\right]^{\tan^2\frac{\theta}{2}}(d\bar{u}^2+2\cos\theta\, dudv+d\bar{v}^2), \tag{I}$$

where b and c are constants and θ is the angle between the asymptotic lines. The surfaces corresponding to (I) are the surfaces of revolution of negative total curvature which have the same spherical representation of their lines of curvature as the catenoid or minimal surface of revolution. To every value of θ from 0 to π, there corresponds a separate surface. Beside the catenoid

corresponding to $\theta = \frac{\pi}{2}$ we may mention a surface generated by revolving the parabola about a line perpendicular to its axis, and also a surface generated by a fourth degree curve having a conjugate point, as being included in the set.

Likewise, if θ_1 is a constant, we have as the first form for the corresponding surfaces

$$d\bar{s}_1^2 = \left[\frac{(1-be^{c(u+v)})^2}{e^{c(u+v)}}\right]^{-\tan^2\frac{\theta_1}{2}} (du^2 + 2\cos\theta_1\, du dv + d\bar{v}^2), \tag{I'}$$

where θ_1 is now the angle between the characteristic lines of course.

These compose the class of surfaces of revolution, of positive total curvature, which have the same spherical representation of their lines of curvature as the catenoid.

CASE 3. An examination of equations (1′) and (2′) shows that if $\theta_1 = \theta$, and the function θ is chosen so that

$$\frac{\partial^2 \log \tan \theta/2}{\partial u^2} = \frac{\partial^2 \log \tan \theta/2}{\partial v^2},$$

the equations which μ and μ_1 must satisfy are one and the same; and, hence the theorem:

To every surface S of negative total curvature which is a solution of the problem and which has the tangent of one-half the angle between the asymptotic lines equal to a function of one of the parameters divided by a function of the other, the lines of reference being lines of curvature, there exists another S_1 of positive total curvature, having the same spherical representation of its characteristic lines as S of its asymptotic.

It is easily shown that any two surfaces S and S_1 which are related in this way are isothermic surfaces, connected by the Bour-Darboux theorem.* It is easily proven: *That all isothermic surfaces S and S_1 connected by the Bour-Darboux theorem are associates of one another, the asymptotic lines of one corresponding to the characteristic lines on the other, and vice versa.*† The converse of this theorem is true.

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* Darboux, "Théorie des Surfaces," Vol. II, p. 243.

† We have discussed in detail in a previous paper the problem of determining such isothermic surfaces.

IV. *Discussion of the Problem from the Standpoint of the Spherical Representation of the Lines of Curvature.*

Taking the Bonnet form* for the fundamental equation of a surface referred to lines of curvature, we have

$$\left.\begin{aligned}\frac{\partial M}{\partial v}-\frac{\partial N}{\partial u}-PQ&=0,\\ \frac{\partial P}{\partial v}+MQ&=0,\\ \frac{\partial Q}{\partial u}-NP&=0,\end{aligned}\right\}\tag{6}$$

where

$$\left.\begin{aligned}M&=-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v},\\ N&=\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u},\end{aligned}\right\}\tag{7}$$

the first and third form for the surface being respectively,

$$d\bar{s}^2=Ed\bar{u}^2+Gd\bar{v}^2,\tag{I}$$
$$d\bar{\sigma}^2=P^2du^2+Q^2dv^2.\tag{III}$$

For the surfaces sought, the parameters being properly chosen, $D=\pm D''$, and hence,

$$\frac{E}{G}=\frac{Q^2}{P^2}\ \text{ or }\ \frac{\sqrt{E}}{\sqrt{G}}=\pm\frac{Q}{P},\tag{8}$$

the plus and minus signs corresponding respectively to surfaces of positive and negative curvature.

Eliminating G from (7) by means of (8), we have

$$\left.\begin{aligned}\frac{\partial\log\sqrt{E}}{\partial v}&=\mp\frac{P}{Q}M,\\ \frac{\partial\log\sqrt{E}}{\partial u}&=-\frac{\partial\log\left(\frac{P}{Q}\right)}{\partial u}\pm N\frac{Q}{P}.\end{aligned}\right\}\tag{9}$$

The condition for the integrability of (9) may be written thus

$$\frac{\partial^2\log\left(\frac{P}{Q}\right)}{\partial u\partial v}\mp\frac{\partial}{\partial u}\left(\frac{MP}{Q}\right)\mp\frac{\partial}{\partial v}\left(N\frac{Q}{P}\right)=0,\tag{10}$$

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* *Journal de l'École Polytechnique*, Vol. XLII, pp. 132–151.

where the upper signs in both (9) and (10) are for surfaces of positive total curvature, the lower for those of negative.

This equation determines the lines on the Gauss sphere which permit of surfaces of positive or negative total curvature of the class sought, having them as the spherical representation of their lines of curvature.

Evidently the two cases reduce to one and the same when $\frac{\partial^2 \log}{\partial u \partial v}\left(\frac{P}{Q}\right) = 0$; that is, when $\frac{P}{Q} = \frac{f(u)}{\phi(v)}$.

The corresponding surfaces are the isothermic surfaces discussed in the preceding article.

If we make the substitution

$$\frac{P}{Q} = \frac{1}{\phi}(\mp \int N\phi\, du + V),$$

(10) transforms into the Laplace equation

$$\frac{\partial^2 \phi}{\partial u \partial v} - \frac{\partial \log M}{\partial u}\frac{\partial \phi}{\partial v} + MN\phi = 0. \tag{11}$$

Eliminating Q from the last two equations of (6) we have again equation (11) with ϕ replaced by P.

The problem, then, is reduced to the simultaneous solution of the following system of equations:

$$\left.\begin{aligned} &\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} - PQ = 0, \\ &\frac{\partial Q}{\partial u} - NP = 0, \\ &\frac{\partial^2 P}{\partial u \partial v} - \frac{\partial \log M}{\partial u}\frac{\partial P}{\partial v} + MNP = 0, \\ &\frac{P}{Q} = \frac{1}{P_1}(\mp \int NP_1\, du + V), \end{aligned}\right\} \tag{12}$$

where P_1 is any solution of the preceding Laplace equation. Given any set of values M, N, P, Q, satisfying this system of equations, we have, by quadrature, from (8) and (9), the functions appearing in the first fundamental form of surfaces which are solutions of the problem. The Cartesian coordinates for any point on the same can be obtained by quadrature as soon as the Cartesian coordinates of the corresponding point on the Gauss sphere are known.

The problem of solving the first three equations of (12) is identical with that of the "Problem of the Spherical Representation" as treated by us in a

previous article.* If we add any fourth equation to these three we can expect, of course, a corresponding solution as the number of equations then only equals the number of functions to be determined. Considering the third equation as a Laplace equation for the determination of P we are led to choose as the new equation some one of the following expressions equated to zero.

$$h = -MN,$$

$$k = -\frac{d^2 \log M}{dudv} - MN,$$

$$k_1 = -\frac{d^2 \log (Mk)}{dudv} - k,$$

$$k_2 = -\frac{\partial^2 \log (M k k_1)}{\partial u \partial v} - k_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$k_i = -\frac{\partial^2 (\log Mk \ldots . k_{i-1})}{\partial u \partial v} - k_{i-1},$$

where i is an arbitrarily large positive integer.

The general solution of the Laplace equation can be found when and only when h or k_i vanishes.† If h vanishes, neglecting the case where the corresponding surfaces are developable, we may say that M vanishes. The corresponding surfaces are those on which the lines v=const. are geodesics. The corresponding lines on the Gauss sphere are great circles. The problem of determining these surfaces, then, is solved as soon as one determines the unit sphere referred to systems of great circles and their orthogonal trajectories.

If, however, h does not vanish but k does, the corresponding surfaces are either surfaces for which one set of the lines of curvature, the lines v=const., have constant geodesic curvature, or certain surfaces having the same spherical representation of their lines of curvature as these.‡ Moreover, all these surfaces can be obtained by algebraic methods and quadrature from the general class corresponding to the case where h vanishes.§

And, in general, the surfaces corresponding to the case $k_{i+1}=0$ can always be obtained by algebra and quadrature from those corresponding to the preceding case $k_i=0$.

The totality of surfaces found in this way, starting with the case $M=0$ and continuing to the case $k_i=0$, where i is arbitrarily large make up the general class belonging to the problem of the spherical representation.

* A. E. Young, "On the Problem of the Spherical Representation," etc., AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, No. 1.

† See previous reference, p. 42. ‡ A. E. Young, *loc. cit.*, p. 47. § *Loc. cit.*, p. 43.

The surfaces of this class belonging to our present problem can be found as follows:

Taking the general class corresponding to the case $M=0$, we choose the corresponding functions E, G, P and Q so as to satisfy the relation *

$$\frac{E}{G}=\frac{UQ^2}{VP^2}, \tag{13}$$

where U and V are functions of u and v, respectively, to be determined. The original class with these limitations on the functions are solutions of the problem, and, in fact, compose that system of surfaces belonging to the problem on which one set of the lines of curvature are geodesics.

Again, taking the general class of surfaces corresponding to $k=0$, we may choose the functions P and Q corresponding, so that the last equation of (12) is satisfied, P_1 now being a general solution of the Laplace equation and hence actually known, and thus obtain at once the surface which we seek from the general class. Or, following the preceding method, we may choose the new functions E, G, P and Q corresponding to the case $k=0$ so that they satisfy (13) and thus obtain the desired surfaces. The surfaces in this case will be those of the problem having one set of lines of curvature characterized by geodesic curvature, and certain other surfaces having the same spherical representation of their lines of curvature as these.

In general, therefore, the surfaces of the problem, involved in the solution of the "Problem of the Spherical Representation," can be obtained from the separate systems as classified above by a process which involves nothing more difficult than the limitation by the simple algebraic equation (13) of the four functions appearing in the first and third fundamental quadratic forms, the lines of reference being lines of curvature.

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* The introduction of the functions U and V is due to the fact that no particular choice of parameters has been made, as was done when equation (8) was derived.

On Orthoptic and Isoptic Loci.

By Prof. Harold Hilton and Miss R. E. Colomb.

Section 1.

The nature of the orthoptic locus (the locus of the intersection of two perpendicular tangents) of a given real plane algebraic curve does not seem to have been investigated with any fulness, except that the degree of the orthoptic locus has been obtained in certain cases and a few of the more elementary properties of the locus have been given, at any rate when the given curve does not pass through the circular points at infinity.*

In the following discussion of the locus we shall denote the degree, class, number of nodes, number of cusps, number of bitangents, number of inflexions, deficiency of the given curve by n, m, δ, $\varkappa$, τ, ι, D; and the corresponding quantities for the orthoptic locus by n', m', δ', $\varkappa'$, δ', ι', D'.

We take ω, ω' as the circular points at infinity. If we choose any points E, F on $\omega\omega'$ so that $(EF, \omega\omega')$ is harmonic, the orthoptic locus will be the locus of the intersection of tangents from E and F to the given curve.

It must be understood throughout that, when we discuss the orthoptic locus of some special type of curve, we imply that the curve is the most general curve of the special type discussed. Otherwise the orthoptic locus may degenerate or be simplified. For instance, the orthoptic locus of a central conic is in general a circle; but it becomes a pair of lines through ω and ω', if the conic divides $\omega\omega'$ harmonically.

It must be understood also that the line $\omega\omega'$ is never counted as part of the orthoptic locus.

Section 2.

First suppose (§§ 2–4) that the given curve has m^2 distinct ordinary foci; *i. e.*, no inflexional tangent or bitangent of the curve goes through ω or ω', the curve does not touch $\omega\omega'$ or go through ω or ω', etc.

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* Bassett, "Elementary Treatise on Cubic and Quartic Curves," §§ 68, 157; Zimmermann, *Crelle* 126 (1903), p. 183; Taylor, *Proc. Royal Soc.*, 37 (1884), p. 138, and *Messenger Math.*, 16 (1886), p. 1; Jonquières, *Nouv. Annales Math.*, 20 (1861); Loucher, *ibid.*, II, 11 (1892); Ljungh, Diss. "Ueber Isoptische und Orthoptische Kurven" (Lund, 1895). We have not been able to peruse this last paper.

Suppose two tangents are drawn to the curve from E and F meeting at P. Let E, and therefore F, approach ω. Then, if the tangents at E and F do not become consecutive, P will approach ω, the pencil $P(EF, \omega\omega')$ being harmonic. Hence, ultimately the orthoptic locus has one branch through ω for each pair of tangents from ω to the curve, and the tangent to the branch and the line $\omega\omega'$ form a harmonic pencil with the pair of tangents from ω to the curve.

Hence, through ω $\frac{1}{2}m(m-1)$ branches of the orthoptic locus pass, and similarly for ω', while there is no other point of the orthoptic locus on $\omega\omega'$. Hence, $n' = m(m-1)$.

Moreover, if ωH, $\omega' H$ and ωK, $\omega' K$ are two conjugate pairs of tangents from ω and ω' (so that H, K are real foci) and C is the middle point of HK, it follows from the harmonic properties of a quadrilateral that $\omega(C\omega', HK)$ and $\omega'(C\omega, HK)$ are harmonic pencils. Therefore, $C\omega$ and $C\omega'$ touch the orthoptic locus at ω and ω', *i. e.*, C is a *singular focus* of the orthoptic locus. Hence,

The orthoptic locus of a curve of class m is in general of degree $m(m-1)$, and its singular foci are the $\frac{1}{2}m(m-1)$ middle points of the segments joining any two of the m real foci of the given curve.

The intersections of a curve with its orthoptic locus are given by the following theorem, the proof of which we leave to the reader:

A curve of degree n and class m in general cuts its orthoptic locus at the $2m$ points of contact of tangents from the circular points to the curve, touches it at the feet of the $m(m+n-4)$ normals which are also tangents to the curve, and cuts it in the $m(m-3)(n-2)$ points from which two perpendicular tangents can be drawn to the curve neither of which coincides with the tangent at the point.

Section 3.

We now consider the *class* of the orthoptic locus. It will be sufficient to find the number of tangents which can be drawn from ω' to the locus.

First consider the tangents from ω' to the locus whose points of contact are not on $\omega\omega'$.

Suppose PH, $P'H$ to be two consecutive tangents to the curve and PK, $P'K$ the consecutive tangents perpendicular to PH, $P'H$. Suppose, moreover, that PP' passes through ω'. Then, because $P(HK, \omega\omega')$ and $P'(HK, \omega\omega')$ are harmonic, HK passes through ω. But H and K are ultimately the points of contact of the tangents PH and PK. Hence,

If a tangent from ω' to the orthoptic locus touches it at a finite point P, the line joining the points of contact of the two perpendicular tangents from P to the curve passes through ω.

Now suppose any line whatever through ω meets the given curve in H and K. Let the tangents at H and K meet at T, and consider the envelope of the line TV where $T(HK, \omega V)$ is harmonic. By what has just been said, every tangent from ω' to the envelope which does not coincide with $\omega\omega'$ will be a tangent to the orthoptic locus. Now, in general, the envelope will not touch $\omega\omega'$, and hence the number of tangents from ω' to the orthoptic locus with a finite point of contact is equal to the class of the envelope.

To find the class of the envelope, we find the number of tangents from ω to the curve. The line TV can not pass through ω unless ωHK touches the given curve at H but not at K (or vice versa). If this is the case, however, ωHK will be a tangent to the envelope at a point U such that $(\omega U, HK)$ is harmonic. Hence, each tangent from ω to the curve is a $(n-2)$-ple tangent to the envelope; so that the class of the envelope is $m(n-2)$.

Now the only tangents from ω' to the orthoptic locus with their points of contact on $\omega\omega'$ are the $m(m-1)$ tangents at the $\frac{1}{2}m(m-1)$-ple point ω' itself. Hence,

$$m' = m(n-2) + m(m-1) = m(m+n-3).$$

Section 4.

We now consider the double points and cusps of the orthoptic locus.

Let PH, $P'H$ be two consecutive tangents to the given curve and PK, $P'K$ the consecutive tangents perpendicular to PH, $P'H$. Then P, P', H, K are concyclic, and therefore the angle $KPP'=$ the angle KHP'. Hence in the limit:

If PH, PK are two perpendicular tangents to a curve, the angle between PK and the tangent to the orthoptic locus at P is equal to the angle KHP.

If PH is a bitangent to the given curve, there are two points of contact H on PH, and therefore there are two tangents to the orthoptic locus at P; *i. e.*, P is a double point of the orthoptic locus. It will be readily seen that, if PH is an inflexional tangent, P is a cusp of the orthoptic locus. Hence,

The orthoptic locus has a double point at each of the $m\tau$ intersections of a bitangent of the given curve with a perpendicular tangent, and has a cusp at each of the $m\iota$ intersections of an inflexional tangent of the given curve with a perpendicular tangent.

It will readily be seen that all the cusps of the orthoptic locus are given in this way. Hence,

$$\varkappa' = m\iota.$$

We have now found n', m', $\varkappa'$ and can deduce δ', τ', ι' from Plücker's equations. We find

$$n' = m(m-1), \quad m' = m(m+n-3), \quad \delta' = \tfrac{1}{2}m\{(m+1)(m-2)^2 + 2\tau\},$$
$$\varkappa' = m\iota, \quad \tau' = \tfrac{1}{2}m\{m(m+n)^2 - (6m^2+6mn+n^2) - m + 22 + 2\delta\},$$
$$\iota' = m(3m+\varkappa-6), \quad D' = \tfrac{1}{2}(m-1)(m-2) + mD.$$

Of the $\frac{1}{2}m\{(m+1)(m-2)^2+2\tau\}$ nodes of the orthoptic locus $m\tau$ are the intersections of the bitangents of the given curve with perpendicular tangents, and $\frac{1}{4}m(m+1)(m-1)(m-2)$ are accounted for by the $\frac{1}{2}m(m-1)$-ple points of the orthoptic locus at ω and ω'. There remain $\frac{1}{4}m(m+1)(m-2)(m-3)$ others. Hence,

There are in general $\frac{1}{4}m(m+1)(m-2)(m-3)$ points from which two pairs of mutually perpendicular tangents can be drawn to a given curve of class m.

The results of §§ 2–4 are illustrated by the conic whose orthoptic locus is a circle through the intersection of the conic and the directrices.

Section 5.

In §§ 5–7 we give very briefly without proof the properties of the orthoptic locus of curves bearing some special relation to the line $\omega\omega'$.

If the given curve touches $\omega\omega'$ at Y, the orthoptic locus has a branch through Z, where $(YZ, \omega\omega')$ is harmonic, corresponding to each tangent from Z to the given curve other than $\omega\omega'$.

We find that, if the given curve touches $\omega\omega'$ at k points (which may or may not be cusps of the curve),

$$n' = (m-k)(m-1), \quad m' = (m-k)(m+n-3-k), \quad \varkappa' = (m-k)\iota.$$

If the given curve has $\omega\omega'$ as inflexional tangent at Y, the branches of the orthoptic locus through Z are all linear and touch $\omega\omega'$ at Z (more generally, if the curve has r-point contact with $\omega\omega'$ at Y, each branch of the orthoptic locus has $(r-1)$-point contact with $\omega\omega'$ at Z), and that

$$n' = (m-1)(m-2), \quad m' = (m-2)(m+n-4), \quad \varkappa' = (m-2)(\iota-1).$$

As examples of the results of §§ 2–5 consider

(i) A parabola, whose orthoptic locus is its directrix.

(ii) $3(x+y) = x^3$, whose orthoptic locus is

$$81y^2(x^2+y^2) - 36(x^2-2xy+5y^2) + 128 = 0.$$

Here we have

$$n = 3, \quad m = 3, \quad \delta = 0, \quad \varkappa = 1, \quad \tau = 0, \quad \iota = 1,$$
$$n' = 4, \quad m' = 4, \quad \delta' = 2, \quad \varkappa' = 2, \quad \tau' = 1, \quad \iota' = 2.$$

The given curve touches $\omega\omega'$ at its cusp $(0, \infty)$, and therefore the orthoptic locus has a double point at $(\infty, 0)$ the tangents there being $9y^2=4$, which are also tangents to the given curve.*

The intersections $(\mp\frac{2}{3}\sqrt{2}, \pm\frac{2}{3}\sqrt{2})$ of the inflexional tangent of the given curve with the perpendicular tangents are cusps of the orthoptic locus.

(iii) $y^2=x^3$, whose orthoptic locus is $729y^2=108x-16$.

The curve has $\omega\omega'$ as inflexional tangent at $(0, \infty)$, and therefore the orthoptic locus touches $\omega\omega'$ at $(\infty, 0)$.

The orthoptic locus meets the curve where $x=-4/9$, which gives the points of contact of the tangents from ω, ω' to the curve. The orthoptic locus touches the curve where $x=2/9$, *i. e.*, at the feet of the two normals of the curve which are also tangents.

Section 6.

In § 5 we considered a curve for which $\omega\omega'$ was a multiple tangent, tacitly assuming that no two of the points of contact divided $\omega\omega'$ harmonically.

If this were the case, the branches of the orthoptic locus at ω and ω' are as in § 5, but the nature of the other points of the locus on $\omega\omega'$ is altered, and the degree of the locus is in general lowered. We shall suppose that $\omega\omega$ touches the curve at Y and Z only, so that $(\omega\omega', YZ)$ is harmonic.

(A) Let the curve have ordinary contact at Y and Z. The locus has $(m-3)$ branches through Y due to two perpendicular tangents to the curve, one of which touches near Z and the other passes close to Y but does not touch near Y or Z. Similarly, it has $(m-3)$ branches through Z. It has also in addition a branch cutting $\omega\omega'$ due to two perpendicular tangents, one of which touches near Y and the other near Z.

Example. The tricuspidal quartic $x^2y^2-4(x^3+y^3)+18xy-27=0$ with orthoptic locus $x+y+2=0$.

(B) Let the curve have ordinary contact at Y and a cusp at Z.

The locus has $(m-3)$ branches through Y and $(m-4)$ through Z, and in addition another branch touching $\omega\omega'$ at Z.

(C) Let the curve have ordinary contact at Y and an inflexion at Z.

The locus has $(m-4)$ branches touching $\omega\omega'$ at Y and $(m-4)$ branches through Z, and in addition a branch touching $\omega\omega'$ at Y.

Example. The curve with tangential equation $\lambda^4+2\mu^4=\lambda^2\mu$ and orthoptic locus $2x^2+y+3=0$.

(D) Let the curve have inflexions at Y and Z.

---

* Because $\omega\omega'$ is a *cuspidal* tangent of the given curve.

The locus has $(m-5)$ branches touching $\omega\omega'$ at Y and similarly for Z. It has in addition a branch touching $\omega\omega'$ elsewhere.

Example. $\lambda^2\mu^2=\lambda^5+\mu^5$ with orthoptic locus

$$(x-y)^2+2(x+y)+4=0.$$

and so on in all similar cases.

Section 7.

We now notice briefly some cases in which the given curve passes through ω and ω'.

If the curve has a linear branch through ω and ω', the orthoptic locus has a branch through ω touching the curve at ω, and a superlinear branch of order 2 at ω for each of the $m-2$ tangents from ω to the curve not touching at ω. The line $\omega\omega'$ and the tangent to the superlinear branch form a harmonic pencil with the tangent at ω and the tangent from ω to the given curve. The orthoptic locus has also a linear branch through ω corresponding to each of the $\frac{1}{2}(m-2)(m-3)$ pairs of tangents from ω to the given curve not touching at ω (see §2). We find

$$n'=m(m-1), \quad m'=m(m+n-5)+4, \quad \varkappa'=2(m-2)+m\iota.$$

The reader may illustrate the above statements on the circle whose orthoptic locus is a concentric circle, or on the cissoid $(x+1)^3+xy^2=0$ whose orthoptic locus is

$$16(x^2+y^2)^3+108(x^2+y^2)(x^2+2y^2)+108x^3+729y^2=0.$$

If the given curve has two linear branches through ω, each branch contributes its quota to the form of the orthoptic locus at ω as stated above. The orthoptic locus has also two superlinear branches of order 2 given by the intersections of two tangents to the curve, one of which touches one branch and the other touches the other branch near ω. The two superlinear branches have a common tangent, and this tangent and $\omega\omega'$ form a harmonic pencil with the tangents to the curve at ω.

Similarly, if the given curve has more than two branches through ω.

If the curve touches $\omega\omega'$ at ω and ω', the orthoptic locus has $m-2$ linear branches through ω, of which $m-3$ touch $\omega\omega'$ at ω in addition to the $\frac{1}{2}(m-3)(m-4)$ branches through ω obtained as in § 2, so that

$$n'=2+4(m-3)+(m-3)(m-4)=(m-1)(m-2).$$

Example. The three-cusped hypocycloid whose orthoptic locus is a circle.

The reader may consider the case in which the given curve has cusps at ω and ω', and illustrate by the cardiode whose orthoptic locus consists of a circle and a limaçon, all three curves having a common singular focus.

Section 8.

If an inflexional or multiple tangent of the given curve passes through ω, this tangent is part of the orthoptic locus. In this section n', m', ... refer to the remainder of the orthoptic locus excluding such tangents.

If a bitangent passes through ω, the orthoptic locus has $2(m-2)$ linear branches through ω touching each other in pairs, and $\frac{1}{2}(m-2)(m-3)$ other linear branches through ω. We have

$$n'=(m+1)(m-2), \quad m'=m(m+n-3)-4, \quad \varkappa'=m\iota.$$

If an inflexional tangent passes through ω, the orthoptic locus has $(m-2)$ superlinear branches of order 2 at ω, and $\frac{1}{2}(m-2)(m-3)$ other linear branches through ω. We have

$$n'=(m+1)(m-2), \quad m'=m(m+n-4), \quad \varkappa'=m\iota-4.$$

Example. $(2x+1)^3=27(x^2+y^2)$ with orthoptic locus

$$(16y^4-16xy^2+4x^2+4y^2-4x-1)(x^2+y^2)=0.$$

If the curve has an inflexion at ω and ω', the inflexional tangents count twice over as part of the locus, and $n'=m(m-1)-4$.

Section 9.

The nature of the *locus of the intersection of two perpendicular normals* of a given curve is readily obtained, for it is the orthoptic locus of its evolute.

Taking the general curve of § 2, its evolute is of degree $n_1=3m+\varkappa$, is of class $m_1=n+m$, has $\iota_1=0$ inflexions, etc. The evolute has n cusps at which $\omega\omega'$ is the tangent.

Hence, putting n_1, m_1, ... for n, m, ... in § 5, and putting $k=n$, we see that the locus of the intersection of two perpendicular normals is of degree $n_1'=m(m+n-1)$, is of class $m_1'=m(4m+\varkappa-3)$, and has $\varkappa_1'=0$ cusps.

If the given curve touches $\omega\omega'$,

$$n_1'=(m-1)(m+n-2), \quad m_1'=(m-1)(4m+\varkappa-6), \quad \varkappa_1'=0.$$

For example, the evolute of a parabola is a semi-cubical parabola whose orthoptic locus is a parabola.

If the given curve has $\omega\omega'$ as inflexional tangent,

$$n_1'=(m-2)(m+n=3), \quad m_1'=(m-2)(4m+\varkappa-9), \quad \varkappa_1'=0.$$

The reader may illustrate on the semi-cubical parabola.

Section 10.

We here enumerate the types of real curve whose orthoptic locus is of the first or second degree.

I. *Orthoptic locus a straight line.*

- (i) Parabola.
- (ii) $n=4$, $m=3$. Curve touches $\omega\omega'$ in two points dividing $\omega\omega'$ harmonically.

II. *Orthoptic locus a circle.*

- (i) Circle.
- (ii) Conic.
- (iii) $n=4$, $m=3$. Curve touches $\omega\omega'$ at ω and ω'.

III. *Orthoptic locus a parabola.*

- (i) $n=3$, $m=3$. Curve has $\omega\omega'$ as inflexional tangent.
- (ii) $n=5$, $m=4$. Curve has $\omega\omega'$ as an ordinary and as an inflexional tangent, the points of contact dividing $\omega\omega'$ harmonically.
- (iii) $n=6$, $m=5$. Curve has two inflexions dividing $\omega\omega'$ harmonically, at which $\omega\omega'$ is the tangent.

IV. *Orthoptic locus a conic.*

- (i) $n=4$, $m=3$. $\omega\omega'$ is a bitangent.
- (ii) $n=6$, $m=4$. $\omega\omega'$ is a triple tangent, two of the points of contact dividing $\omega\omega'$ harmonically.
- (iii) $n=8$, $m=5$. $\omega\omega'$ is a quadruple tangent, two pairs of points of contact dividing $\omega\omega'$ harmonically.

As an example the reader may enumerate the fourteen types of curve whose orthoptic locus is a cubic, and the thirty-eight types whose orthoptic locus is a quartic.

Section 11.

We now state the properties of the *isoptic locus* of a given curve; *i. e.*, the locus of the intersection of two tangents inclined at any given angle α.*

To each pair of tangents to the curve from ω correspond two branches of the isoptic locus through ω. We have

$$n'=2m(m-1), \quad m'=2m(n+m-2), \quad \varkappa'=2m\iota.$$

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* We suppose $\alpha \not\equiv 0$ and $\alpha \not\equiv \frac{1}{2}\pi$. It is not in general possible to distinguish the case in which two tangents cut at an angle α, from that in which they cut at an angle $\pi-\alpha$. But consider the case of the circle or parabola.

The curve cuts the isoptic locus at the $2m(m-3)(n-2)$ points from which two tangents can be drawn inclined at an angle α and neither of them touching at the point. It touches the locus at the $2m$ points of contact of tangents from ω and ω' * and at the $2m(m+n-4)$ points such that a line making an angle α with the tangent at the point touches the curve elsewhere.

If the curve touches $\omega\omega'$ at Y, the locus has $(m-1)(m-2)$-ple points at ω and ω', and has an $(m-1)$-ple point at Z where $(Y\omega\, Z\omega')$ has the cross-ratio $e^{\pm 2i\alpha}$. We find

$$n'=2(m-1)^2, \quad m'=2(m-1)(m+n-3), \quad \varkappa'=2(m-1)\iota.$$

If the curve has $\omega\omega'$ as inflexional tangent at Y, the locus has $(m-2)(m-3)$-ple points at ω and ω' and has $(m-2)$ linear branches touching $\omega\omega'$ at Z. We find

$$n'=2(m-1)(m-2), \quad m'=2(m-2)(m+n-3), \quad \varkappa'=2(m-2)(\iota-1).$$

As examples consider (we write k for $\tan^2\alpha$):

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with isoptic locus $k(x^2+y^2-a^2-b^2)^2=4a^2b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$.

(ii) $y^2=4ax$ with isoptic locus $y^2-4ax=k(x+a)^2$.

(iii) $y^2=x^3$ with isoptic locus

$$3^{12}k(x^2-ky^2)^2-3^9 2^3\{(k^2+k+2)x^3+(k^3+3k^2)xy^2\}$$
$$+3^6 2^4\{(k^3+4k^2+9k)x^2+(2k^3+27k^2+54k+27)y^2\}-3^3 2^7(k^3+3k^2)x+2^8k^3=0,$$

touching the curve where $x=-4/9$ and $k(9x-2)^2=81x$, and touching $\omega\omega'$ where $x^2=ky^2$.

To find the nature of the locus of the intersection of two tangents, one drawn to each of two given curves and inclined at a constant angle, we have only to take away the isoptic locus of each curve considered separately from the isoptic locus of the (degenerate) curve, consisting of the two curves taken together.†

The case in which the given curves are both circles is well known.

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* Contrast with the orthoptic locus.

† Taylor, *loc. cit.*

A New Canonical Form for Systems of Partial Differential Equations.

By L. B. Robinson.

Introduction.

Some years ago Delassus obtained a canonical form useful for the study of systems of partial differential equations.* It is possible to reduce to this form a very large class of differential systems, but that the form is not absolutely general was discovered almost simultaneously by the author † and by M. Gunter.‡ Were the integration of the given differential system the only question of interest there would be no need to improve the canonical form given by Delassus, for when the given differential system is of the first order the above-mentioned canonical form is always valid, and it has been shown by Riquier that it is always possible to reduce a differential system to the first order.§

But since canonical forms are often useful in the study of the comitants of either differential or algebraic systems, it seems desirable to give a canonical form which has no exceptional cases and will apply equally to a differential system or a system of polynomials. In the following paper the author will construct such a canonical form. It will be noted that the symbols $X_1, X_2 \dots$ which are used to represent differential operators could likewise be interpreted as the variables of a system of polynomials.‖. As the new canonical form is closely analogous to that of Delassus, it seems that it will be equally useful in studying the invariants or the characteristics of differential systems.¶ In fact this turns out to be the case in some simple cases considered by the author.

* *Annales de l'école normale supérieure*, Vol. XIII, 3d Ser., p. 421.

† "Notes from the Mathematical Seminary," J. H. U., 1913. *Comptes Rendus*, 15 Juillet 1913.

‡ *Comptes Rendus*, 14 Avril 1913.

§ Riquier, "Sur les systèmes d'équations aux dérivées partielles," *Annales de l'école normale*, Vol. X, 3d Ser., p. 359.

‖ Delassus, *Annales de l'école normale supérieure*, Vol. XIV, 3d Ser., p. 21.

¶ For further bibliography consult Gunter, "On the Theory of the Characteristics of Systems of Partial Differential Equations" (Russian), *Comptes Rendus*, 13 Octobre 1913; 23 Mars 1914; 20 Avril 1914. Janet, *Comptes Rendus*, 13 Janvier 1913; 27 Octobre 1913. In the three above-mentioned notes of M. Gunter in the *Comptes Rendus*, results somewhat similar to those of the author seem to have been obtained.

§ 1. *A Property of Differential Systems with Regular Initial Conditions.*

Consider a system S of partial differential equations with independent variables $x_1, x_2, \ldots, x_q$ and unknown functions $u_1, u_2, \ldots, u_s$. The equations are supposed to be solved for different derivatives of the u's in such a way that no second member can contain a derivative of order higher than that of the corresponding first member. A derivative which is a first member of S, or a derivative obtained from a first member by differentiation is called a principal derivative. All others are named parametric.* Let the values of the unknowns u and of a suitably chosen number of their parametric derivatives be given as functions of some of the variables x for certain initial values of the remaining variables x. We shall suppose that these initial conditions are all regular in the sense of Riquier.† Let an integer or 'cote' be associated with each of the variables x and the unknowns u in some convenient way,‡ the 'cotes' of the x's being unity, and let the 'cote' of a derivative of an unknown u_p be obtained by adding the order of the derivative to the 'cote' of u_p.

Let Γ be the maximum 'cote' of the first members of the initial conditions. We shall now consider the group of equations $S_{\Gamma+1}$ obtained from the system S, prolonged by repeated differentiations, by selecting those equations in which the 'cote' of the first members is $\Gamma+1$. To simplify writing we shall consider derivatives with respect to four independent variables only as the reasoning would be the same in the general case, and shall use the following abbreviation for a derivative of a typical variable u

$$\frac{\partial^{n_1+n_2+n_3+n_4}u}{\partial x_1^{n_1}\partial x_2^{n_2}\partial x_3^{n_3}\partial x_4^{n_4}}=(n_1,\, n_2,\, n_3,\, n_4).$$

We shall now examine the three derivatives

$$\begin{aligned}\alpha &= (n_1,\, n_2,\, n_3,\, n_4),\\ \beta &= (n_1,\, n_2,\, n_3+1,\, n_4),\\ \gamma &= (n_1,\, n_2,\, n_3+1,\, n_4-1).\end{aligned}$$

Suppose that the first of these is a principal derivative occurring in the group $S_{\Gamma+1}$. The second is obviously obtained from the first by differentiation. Hence, if the first is principal, so is the second. It is to be shown, assuming as we have done, that the initial conditions are regular, that γ is also principal.

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* Riquier, "Les systèmes d'équations aux derivées partielles, Chapter VI.

† *Loc. cit.*, Chapter XII.

‡ Riquier, Chapter VII.

For suppose γ to be parametric. Then it is derived from some parametric derivative of 'cote' Γ, obtained by subtracting unity from the exponent of one of its independent variables. Hence there are four possible primitives of γ, viz.:

$$(n_1, n_2, n_3+1, n_4-2) \quad =\phi_1(x_2, x_3, x_4, x_5 \ldots .), \tag{1}$$

$$(n_1, n_2, n_3, n_4-1) \quad =\phi_2(x_2, x_3, x_4, x_5 \ldots .), \tag{2}$$

$$(n_1, n_2-1, n_3+1, n_4-1) =\phi_3(x_2, x_3, x_4, x_5 \ldots .), \tag{3}$$

$$(n_1-1, n_2, n_3+1, n_4-1) =\phi_4(x_2, x_3, x_4, x_5 \ldots .). \tag{4}$$

We have written the above initial conditions in the most general form. For in the ϕ's one of the variables, to fix ideas say x, can not occur as an argument because if it did no derivatives of u could be principal, therefore u would be a parameter just as in the following system:

$$\frac{\partial v}{\partial x}=\phi(x, y, u), \quad \frac{\partial w}{\partial x}=\psi(x, y, u).$$

In the above system u is an arbitrary unknown. Let us write $u=\theta(x, y)$ where θ is an arbitrary function of x and y. The derivatives of u are obtained by differentiating θ. They are evidently parametric.

As for $x_2, x_3, \ldots .$, we do not assert that they all occur actually in the ϕ's, but when we say the system has regular initial conditions we mean that if x_i occurs actually in one of the functions ϕ then $x_{i+1}, x_{i+2}, \ldots ., x_q$ also occur actually in the same ϕ.

Now returning to the equations (1), (2), (3), (4) we observe first of all that x_4 can not occur actually in (1), for if it did β could not be a principal derivative. For the same reason x_3 can not appear actually in (2) nor x_2 actually in (3). And, therefore, from the hypotheses that the initial conditions are regular, it follows that x_2 and x_3 do not occur in (1), and x_2 does not occur in (2).

So then we see that we can not obtain γ from the group of initial conditions under the hypothesis that they have the regular form. Therefore γ must be a principal derivative. In exactly the same way we can show that the two derivatives

$$(n_1+1, n_2, n_3, n_4-1),$$
$$(n_1, n_2+1, n_3, n_4-1),$$

must be principal also. We may now announce the following result:

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Suppose given a system S where the initial conditions are regular. Let Γ be the maximum 'cote' of the first members of the initial conditions. Let

$$(n_1, n_2, \ldots, n_p)$$

be a typical first member of the group $S_{\Gamma+1}$. Then all derivatives obtained by differentiating the above expression once, with respect to one of the independent variables $x_1, x_2, \ldots, x_p$ which are engaged therein, and then reducing by unity the exponent of the last independent variable involved, are also principal.

Call this property "K." As a simple example we may suppose $\frac{\partial^2 u}{\partial x_2^2}$ to be a first member of 'cote' $\Gamma+1$ belonging to a system S involving only two independent variables x_1 and x_2. Differentiate with respect to both of these independent variables. We obtain the derivatives

$$\frac{\partial^3 u}{\partial x_1 \partial x_2^2}, \quad \frac{\partial^3 u}{\partial x_2^3}.$$

Then, if the initial conditions are regular, the group of first members of $S_{\Gamma+1}$ must also contain $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ as well as $\frac{\partial^2 u}{\partial x_2^2}$.

§ 2. *Proof that Systems with the Property K Are Regular.*

Firstly, we shall show that if the given system has the property K, the prolonged system has this property also. For, let

$$(n_1, n_2, \ldots, n_q)$$

be a derivative belonging to the group of first members of the system non-prolonged. Differentiate an arbitrary number of times with respect to each variable and we obtain

$$(n_1+\lambda_1, n_2+\lambda_2, \ldots, n_q+\lambda_q).$$

Differentiate this last derivative with respect to x_l and integrate once with respect to x_q. We obtain

$$(n_1+\lambda_1, n_2+\lambda_2, \ldots, n_l+\lambda_l+1, \ldots, n_q+\lambda_q-1),$$

and this last expression can clearly be derived from

$$(n_1, n_2, \ldots, n_l+1, \ldots, n_q-1),$$

while this last derivative surely exists among the group of first members if we assume that the system has the property K.

Select one of the unknowns involved, for example u, and arrange all its derivatives whatsoever whose 'cote' is Γ in the order* adopted by Delassus, and call this set of derivatives (A). From now on, following the example of Delassus, we shall use the notation

$$X_1^{a_1}, X_2^{a_2}, X_3^{a_3}, \ldots, X_p^{a_p} \equiv \frac{\partial^{a_1+a_2+a_3, \ldots, a_p} u}{\partial x_1^{a_1}, \partial x_2^{a_2}, \partial x_3^{a_3}, \ldots, \partial x_p^{a_p}}.$$

Let $X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}$, which we shall call (a), be a member of the group (A). Differentiate with respect to x_1 and we obtain

$$X_1^{n_1+1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}.$$

Among the group of the derivatives of u there are a certain number from which we can obtain the last one by a single differentiation. One of these is

$$X_1^{n_1+1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p-1},$$

which is anterior to (a) in the set (A). Differentiate (a) with respect to x_2 and we obtain

$$X_1^{n_1}, X_2^{n_2+1}, X_3^{n_3}, \ldots, X_p^{n_p},$$

which can be obtained from

$$X_1^{n_1}, X_2^{n_2+1}, X_3^{n_3}, \ldots, X_p^{n_p-1},$$

which belongs to the set (A). And continuing thus we can show that any derivative of (a) taken once with respect to $x_1, x_2, \ldots, x_{p-1}$ can be obtained by differentiating some member anterior to (a) with respect to x_p if $n_p \geqq 1$, which is the case by hypothesis.

Now differentiate twice with respect to x_1 and we obtain

$$X_1^{n_1+2}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{n_p}.$$

If $n_p > 1$ this can be obtained from $X_1^{n_1+1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}-2}$ by differentiating twice with respect to x_p. If $n_p = 1$ it can be obtained from

$$X_1^{n_1+2}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}-1}$$

by differentiating with respect to x_{p-1}, x_p. This course of reasoning is general and we can say:

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* Delassus, "Extension du Théorème de Cauchy aux systemès les plus generaux des équations aux dérivées partielles," *Annales de l'école normale supérieure* (1896), Vol. IX.

Let (A) be the totality of the derivatives of u with respect to $x_1, x_2, \ldots, x_q$ which have the 'cote' Γ. Let

$$(a) = X_1^{n_1}, \ldots, X_p^{n_p}, \qquad (n_p \geqq 1)$$

be a member of the set. If we differentiate (a) once with respect to one of the variables $x_p, x_{p+1}, \ldots, x_q$, and perform similar differentiations upon the other elements of (A), the totality of derivatives found includes all the derivatives of u of 'cote' $\Gamma+1$. By differentiating similarly K times, a set is formed which includes all derivatives of 'cote' $\Gamma+K$.

Suppose now that our system has the property K, and consider again a derivative of the set (A):

$$(a) = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}.$$

If it is principal, all its derivatives are principal. If it is parametric, we make the following argument:

Differentiate with respect to $x_p, x_{p+1}, \ldots, x_q$. We obtain

$$\left.\begin{array}{l} X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p+1}, \\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}, X_{p+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}, X_{p+j}, \end{array}\right\} \qquad \text{(I)}$$

$$\left.\begin{array}{l} X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}, X_{p+j+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_p^{n_p}, X_q. \end{array}\right\} \qquad \text{(II)}$$

If any one of the derivatives belonging to (I) or (II) is principal, then any preceding one is so, on account of the property K, which is assumed to hold good for our system. Hence there are three possibilities:

(α) All derivatives obtained from (a) by a single differentiation with respect to $x_p, x_{p+1}, \ldots, x_q$ are principal.

(β) They are all parametric.

(γ) They are divisible into sets (I) and (II); of which the first is composed of principal derivatives, the second of parametric.

Since (I) is composed of principal derivatives, any expression obtained from it by differentiation is also principal. As for the derivatives of (II), any one obtained by operating with $x_{p+j+1}, x_{p+j+2}, \ldots, x_q$ only may or may not be principal. But if we differentiate with respect to any members of the set $x_1, x_2, \ldots, x_{p+j}$ only, once or a greater number of times, we obtain a derivative which is principal because we could obtain it from (I) also.

We shall try to satisfy the requirements by initial conditions of the fol-

lowing types. If a certain group of first members extracted from our systems satisfies condition (α), we shall write down as initial condition corresponding to this type

$$(a) = X_1^{n_1},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_p^{n_p} = a \text{ constant.}$$

If condition (β) is satisfied, we shall write down as corresponding initial condition,

$$(a) = X_1^{n_1},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_p^{n_p} = \phi(x_p,\ x_{p+1},\ \ldots,\ x_q).$$

Finally, if condition (γ) is satisfied, we shall write the initial condition thus:

$$(a) = X_1^{n_1},\ X_2^{n_2},\ X_3^{n_3},\ \ldots,\ X_p^{n_p} = \phi(x_{p+j+1},\ x_{p+j+2},\ \ldots,\ x_q).$$

Note that $x_1,\ x_2,\ \ldots,\ x_{p-1}$ are excluded from ϕ in all three cases. In fact we agree that this exclusion shall make part of our hypotheses.

Having assumed that the above types of initial conditions probably satisfy requirements, we shall now proceed to demonstrate rigorously that our supposition is correct.

We proceed as follows confining our attention to the case (γ), because as a result of the exclusion of $x_1,\ x_2,\ \ldots,\ x_{p-1}$ from the ϕ, the argument would not be altered if conditions (α) or (β) were satisfied.

If we differentiate (a) we shall obtain the initial conditions corresponding to (II) and its derivatives with respect to $x_{p+j+1},\ x_{p+j+2},\ \ldots,\ x_q$. The right-hand members are

$$\frac{\partial\phi}{\partial x_{p+j+1}},\ \frac{\partial\phi}{\partial x_{p+j+2}},\ \ldots,\ \frac{\partial\phi}{\partial x_q},$$

and their derivatives with respect to the variables involved effectively. Hence it is clear that if ϕ contains actually all the variables in the range

$$x_{p+j+1},\ x_{p+j+2},\ \ldots,\ x_q,$$

we should certainly obtain by prolongation all the initial conditions corresponding to the parametric derivatives of 'cote' higher than Γ. The only questions which can now arise are the following:

(1) Can any derivative of the ϕ's correspond to a principal derivative? For in that case the ϕ's could not possibly contain all the variables of the range just mentioned actually.

(2) Can we, by prolonging the set of initial conditions constructed in the manner described in the preceding paragraphs,* obtain two different expressions for the same parametric derivative?

* The description of the construction of the initial conditions runs from the top of p. 100 to the bottom of p. 101.

If we can answer these two questions in the negative, we shall have demonstrated that the initial conditions constructed in the way just described, and which are obviously regular, correspond exactly to the type of system under consideration, *i. e.*, systems with the property K.

Solution of the First Question.

Firstly, we agree that from now on systems which have the property K shall be called canonical. Then let

$$\alpha = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s}$$

be a derivative whose 'cote' is Γ, where

$$n_1+n_2+n_3+\ldots+n_{p-1}+S=n-1.$$

Furthermore, let

$$\beta = X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-1}^{m_{p-1}}, X_p^{s+i}$$

be a derivative whose 'cote' is $\Gamma+1$, where

$$m_1+m_2+m_3+\ldots+m_{p-1}+S+i=n,$$

and where i is an integer positive or negative.

Suppose that the first of these two quantities is parametric. In fact, were it not so, it would not figure among the group of derivatives corresponding to arbitrary initial conditions, and we should not be obliged to consider it at all. Our object is to prove that if we differentiate the expression

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s} = \phi(x_p, x_{p+1}, \ldots, x_q)$$

with respect to $x_p, x_{p+1}, \ldots, x_q$, we shall not obtain any derivative identical with β or a derivative of β.*

To accomplish this we proceed as follows: Firstly, suppose one of the m's to be greater than the n with the same subscript. We then write side by side

$$(\alpha)\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s} = \phi(x_p, x_{p+1}, \ldots),$$
$$(\beta)\ X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-1}^{m_{p-1}}, X_p^{s+i} = M.$$

If we differentiate the first of these with respect to $x_p, x_{p+1}, \ldots, x_q$ only, the exponents of $X_1, X_2, X_3, \ldots, X_{p-1}$ will not be increased, and therefore one of the m's will always be superior to the n with the same subscript. Therefore,

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* It may happen that $x_p, x_{p+1}, \ldots, x_{p+j}$ do not occur actually in ϕ. Then we differentiate with respect to $x_{p+j+1}, x_{p+j+2}, \ldots, x_q$.

in this case, a derivative of (α) with respect to the specified variables, can not be identical with β or a derivative of β. But if

$$n_r \geqq m_r, \qquad (r=1, 2, \ldots, p-1)$$

for all values of r, we differentiate β as many times as is necessary to make the exponent of X_1 equal to n_1, the exponent of X_2 equal to n_2, and in general the exponent of X_r equal to n_r, where $r=1, 2, \ldots, p-1$. Since $i \geqq 1$, we may now differentiate (α) with respect to x_p until s becomes equal to $s+i$. Now we can write the two expressions derived from α and β,

$$\alpha' = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s+i} = \phi_{x_p}^{(i)}(x_p, x_{p+1}, \ldots),$$
$$\beta' = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s+i} = M^{(i-1)}.$$

Now if ϕ contains x_p actually, α' is parametric, while β' is principal, since it is derived from β which is principal. But $\alpha'=\beta'$, therefore if $\phi_{x_p}^{(i)}$ contains x_p actually, we are led to a contradiction. But we shall now show that x_p can not occur actually in ϕ which is the primitive function from which $\phi_{x_p}^{(i)}$ is derived, if the system has the property K.

For we have assumed that

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-1}^{m_{p-1}}, X_p^{s+i}$$

is principal. Therefore,

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+1}, X_p^{s+i}$$

is also principal. Then from the property K it follows that

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+1}, X_p^{s+i-1}$$

is principal. A repeated application of the same rule shows us that

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+2}, X_p^{s+i-2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^{s+1},$$

are all principal. Now let us consider the group derived from

$$\alpha = X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s} = \phi(x_p, x_{p+1}, \ldots),$$

by differentiating it once with respect to $x_p, x_{p+1}, \ldots, x_q$. We write it

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s+1},$$
$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s}, X_{p+1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{s}, X_q.$$

The first of these we have just shown to be principal, but in any case it follows from the fact that the first member of the above group is principal that we

must equate α to $\phi(x_{p+1}, x_{p+2}, \ldots, x_q)$, where ϕ may not contain actually all the arguments written down, but surely contains all those posterior to the first one which occurs actually. Then, if we differentiate α with respect to $x_{p+1}, x_{p+2}, \ldots, x_q$, we shall not obtain any derivative identical with β. For we have assumed in this case that $n_i \geqq m_i$ for all values of i. Also the order of β is greater than the order of α by unity. Hence, the exponent of X_p in β is greater than the corresponding exponent in α. And if we prolong α by differentiating with respect to $x_{p+1}, x_{p+2}, \ldots, x_q$, we shall not increase the exponent of X_p. Therefore by that operation we can never obtain a derivative identical with β or with a derivative of β.

The above is a special case given to illustrate the method employed. Let us pass to the consideration of the most general case. Let us compare

$$(\gamma)\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^{s},$$
$$(\delta)\ X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}}, X_p^{r}, X_{p+\lambda}^{t}, X_{p+\lambda+1}^{j}, \ldots, X_q^{K},$$

where

$$n_1+n_2+n_3+\ldots+n_{p-2}+n_{p-1}+S=n-1,$$

and

$$m_1+m_2+m_3+\ldots+m_{p-2}+m_{p-1}+r+t+j+\ldots+K=n.$$

Assume that (γ) is parametric and (δ) principal. In the case where at least one of the m's is greater than the n with the same subscript, it is obvious that no derivative of (γ) with respect to $x_p, x_{p+1}, \ldots, x_q$ is identical with (δ) or a derivative of (δ). But, suppose,

$$n_i \geqq m_i, \qquad (i=1, 2, \ldots, p-1).$$

There are two cases to be disposed of. Firstly, let $r>S$. Since (δ) is principal,

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+1}, X_p^{f}, X_{p+\lambda}^{t}, X_{p+\lambda+1}^{j}, \ldots, X_q^{K}$$

is surely principal, and it follows from the property K that

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}+1}, X_p^{f}, X_{p+\lambda}^{t}, X_{p+\lambda+1}^{j}, \ldots, X_q^{K-1}$$

is principal. And a repeated application of the above process shows us that

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^{s+1}$$

is certainly principal. Hence x_p can not occur actually in the arbitrary function ϕ which corresponds to (γ). And since $r>S$ no derivative of (γ) with respect to $x_{p+1}, x_{p+2}, \ldots, x_q$ is identical with (δ) or with a derivative of (δ).

Now let us consider the case where $r \leqq S$. Then

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_{p-2}^{m_{p-2}}, X_{p-1}^{m_{p-1}}, X_p^r, X_{p+1}, X_{p+\lambda}^t, \ldots, X_q^{K-1}$$

is principal and continuing the process, we show that

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^s, X_{p+1}$$

is principal. In a similar fashion we can show that

$$\begin{gathered} X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^s, X_{p+2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-2}^{n_{p-2}}, X_{p-1}^{n_{p-1}}, X_p^s, X_{p+\lambda}, \end{gathered}$$

are principal, and we write our initial condition

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, \ldots, X_{p-1}^{n_{p-1}}, X_p^s = \phi(x_{p+\lambda+1}, \ldots).$$

Comparing this with

$$X_1^{m_1}, X_2^{m_2}, X_3^{m_3}, \ldots, X_q^K = M,$$

and prolonging the two equations, we shall arrive at no incompatibility since the second expression contains a factor $X_{p+\lambda}^f$ where $f \neq 0$. Thus we have given a negative response to the first question.

Solution of the Second Question.

Let us write

$$\left.\begin{aligned} X_1^{n_1}, X_2^{n_2}, \ldots, X_p^{n_p} &= \phi(x_p, x_{p+1}, \ldots), \quad n_p \neq 0, \\ X_1^{m_1}, X_2^{m_2}, \ldots, X_p^{m_p} &= \psi(x_p, x_{p+1}, \ldots), \quad m_p \neq 0, \end{aligned}\right\} \qquad \text{(III)}$$

where

$$\begin{aligned} n_1 + n_2 + \ldots + n_p &= n-1, \\ m_1 + m_1 + \ldots + m_p &= n-1. \end{aligned}$$

At least one of the m's differs from the corresponding n, for the two expressions are assumed to be different. Then since the sum of the m's equals the sum of the n's for one value of i at least, $m_i > n_i$, and for at least one value of j, which letter is different from i, we have the inequality $n_j > m_j$. It follows that the exponents of at least two of the X's in each of the expressions of (III) are different. Therefore, we have the inequality $n_i \neq m_i$ where $i \neq p$. Therefore, if we differentiate (III) an arbitrary number of times with respect to x_p, $x_{p+1}, \ldots, x_q$, we shall never arrive at incompatibilities. In case $m_p = 0$, $n_p \neq 0$, the course of reasoning is not different, for we write

$$\begin{gathered} X_1^{n_1}, X_2^{n_2}, \ldots, X_{p-1}^{n_{p-1}}, X_p^{n_p} = \phi(x_p, x_{p+1}, \ldots), \\ X_1^{m_1}, X_2^{m_2}, \ldots, X_{p-1}^{m_{p-1}} = \psi(x_{p-1}, x_p, \ldots). \end{gathered}$$

Then at least one of the m's is greater than the corresponding n, and again we see that no incompatibility can arise.

All that we have now proven is only a generalization of section 197 of Riquier's book.

Section 3.

The facts which are now at our disposal will be used to rectify a defect in the work of Delassus. This author, in the paper we have cited,* has reduced any set of equations to a canonical form of which, however, there are exceptional cases. The determinant

$$\Delta = \frac{\partial(\xi_1, \xi_2, \ldots, \xi_{p-1}, \zeta_1)}{\partial(\xi'_1, \xi'_2, \ldots, \xi'_p)},$$

mentioned on page 436 of his work actually does vanish for certain exceptional cases. The following example makes this clear. Indeed, the reasoning of the last paragraphs of page 436, and the first paragraphs of the following page is a little defective.

It is easy to show that if we are given a set of partial differential equations, there is no unique canonical form which is at the same time orthonomic.

To prove this let us take a system of partial differential equations of the second order containing three equations. When there are only one unknown and three independent variables, the maximum number of derivatives which may be involved in this system is six. Denote these by X_1^2, X_1X_2, X_1X_3, X_2^2, X_2X_3, X_3^2, and take them three at a time. There will be twenty different combinations.

Let us suppose that X_1^2, X_1X_2, X_1X_3 are the first members of our system. In other words let us write the system thus:

$$X_1^2 = f_1, \quad X_1X_2 = f_2, \quad X_1X_3 = f_3, \tag{I}$$

the second members being functions of the independent variables x_1, x_2 and x_3 only. Let us now consider the system

$$X_1^2 = \phi_1, \quad X_1X_2 = \phi_2, \quad X_2^2 = \phi_3, \tag{II}$$

the second members being of the same type as those of the first system. Furthermore, let them satisfy the conditions for the passivity of system (II). We can easily show that no linear transformation of variables will send the set (II) into a set of type (I). For, make the change,

$$\begin{aligned} x'_1 &= ax_1 + bx_2 + cx_3, \\ x'_2 &= a'x_1 + b'x_2 + c'x_3, \\ x'_3 &= a''x_1 + b''x_2 + c''x_3. \end{aligned}$$

* *Annales de l'école normale supérieure*, Vol. IX (1896).

Then,

$$X_1^2 = a^2 X_1'^2 + 2aa' X_1' X_2' + 2aa'' X_1' X_3' + \ldots,$$
$$X_1 X_2 = ab X_1'^2 + (ab' + a'b) X_1' X_2' + (ab'' + a''b) X_1' X_3' + \ldots,$$
$$X_2^2 = b^2 X_1^2 + 2bb' X_1' X_2' + 2bb'' X_1' X_3' + \ldots,$$

and it is easy to see that the determinant

$$\begin{vmatrix} a^2, & 2aa', & 2aa'', \\ ab, & ab'+a'b, & ab''+a''b, \\ b^2, & 2bb', & 2bb'', \end{vmatrix}$$

vanishes.

We next consider the possibility of transforming (II) into a system whose first members are X_1^2, X_1X_2, X_2X_3. We easily see that no choice of 'cotes' will make these three quantities normal to all three of the expressions X_1X_3, X_2^2, X_3^2, and therefore a change of variables followed by a resolution, leads to a set which is non-orthonomic. And if we examine the eighteen cases which remain, we shall see that we shall obtain similar results each time.

The above simple examples show us that cases exist where it is impossible to make a reduction to a fixed canonical form which is also orthonomic.

Section 4.

In what follows we shall show that it is always possible, by the aid of a linear homogeneous transformation of the independent variables, to put any system of equations into a form which shall be a special case of what we have called the canonical form. This form, indeed, varies with the system of equations, but is so defined that when the original system is given, we can say what the canonical form shall be. Likewise we shall see that we can determine whether or not the system is passive by a finite number of operations, since our canonical form is regular. From now on we shall call the canonical form, defined in the first section, form (A), and the new form about to be defined, form (B). (A) contains (B), that is if a system enters into form (B), it certainly belongs also to (A).

We shall, to simplify writing, define our new form in the case of three independent variables only. It is characterized by the following property:

If $X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3}$ *is a first member of the set, then*

$$X_1^{\alpha_1} X_2^{\alpha_2+1} X_3^{\alpha_3-1}, \quad X_1^{\alpha_1+1} X_2^{\alpha_2} X_3^{\alpha_3-1}, \quad X_1^{\alpha_1+1} X_2^{\alpha_2-1} X_3^{\alpha_3},$$

are also first members. In fact any quantity is a first member which possesses the double property of

(*a*) *being obtained from* $X_1^{a_1}X_2^{a_2}X_3^{a_3}$ *by adding unity to one exponent, and subtracting it from another.*

(*b*) *being anterior to* $X_1^{a_1}X_2^{a_2}X_3^{a_3}$ *according to the definition of anterior given by Delassus.*

Let us call this double property (J). It is easy to see that this new form is less general than the canonical form (A). For if a system possesses the property (K), the existence of $X_1^{a_1}X_2^{a_2}X_3^{a_3}$ among the group of first members involves the existence of $X_1^{a_1}X_2^{a_2+1}X_3^{a_3-1}$ and $X_1^{a_1+1}X_2^{a_2}X_3^{a_3-1}$ in the same group, but not the existence of $X_1^{a_1+1}X_2^{a_2-1}X_3^{a_3}$.

In the second section we have shown that if a system possesses the property (K), the prolonged system possesses the same property. By the same method we can show that if a system possesses the property (J) the prolonged system possesses it also.

Section 5.

We shall now proceed to develop a method by which we can transform the most general system of partial differential equations into canonical form (B). Firstly, we shall demonstrate the following lemma:

Given an equation

$$\delta=f(x, y, \ldots, \sigma, \ldots, \tau, \ldots),$$

belonging to the most general system S, where τ and σ represent the derivatives of a class greater and less than δ, respectively. Suppose that it is desirable to solve the system according to the general method employed for the solution of implicit functions, and that the initial values are $x_0, y_0, \ldots, \sigma_0, \ldots, \tau_0, \ldots, \delta_0$. It is always possible to solve the system S in such a way that the initial values of all derivatives of the type $\frac{\partial f}{\partial \tau}$ shall vanish.*

For, consider the system

$$\delta_1=f_1(x, y, \ldots, \sigma, \ldots, \tau, \ldots), \delta_2=f_2, \ldots, \delta_n=f_n.$$

Suppose that $\frac{\partial f_1}{\partial \tau_1} \neq 0$. Then we can solve the first equation of the system with respect to τ_1 and eliminate τ_1 from the second members of the remaining equations of the set.† Since τ_1 does not occur among the first members of these remaining equations, nor $\delta_2, \delta_3, \ldots, \delta_n$ among the arguments of the first

* For the meaning of this word see Riquier, *loc. cit.*, p. 208.

† τ_1 is any one of the τ's.

equation, or of the first equation solved with respect to τ_1 the system can now be written:

$$\tau_1=\bar{f}_1(xy, \ldots, \delta_1, \sigma, \ldots, \tau, \ldots), \delta_2=\bar{f}_2, \delta_3=\bar{f}_3, \ldots, \delta_n=f_n.$$

And so we can repeat this process until we exhaust all possibilities and obtain a system of the type desired.

We will now suppose, in order to fix ideas, that our system is composed of equations all of order n, and involving only one unknown u. Furthermore, let it be of the type described in the lemma. Finally we shall write it

$$S\begin{cases} \xi_1 \quad +\phi_1 \quad (\eta \ldots)=0, \\ \xi_2 \quad +\phi_2 \quad (\eta \ldots)=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ \xi_{p-1}+\phi_{p-1}(\eta \ldots)=0, \\ \zeta \quad +\phi_p \quad (\eta \ldots)=0. \end{cases}$$

If we suppose all the derivatives of order n written down in the order adopted by Delassus,* ξ_2 shall be supposed posterior to ξ_1, ξ_3 posterior to ξ_2, etc. But we do not assert that ξ_2 follows immediately after ξ_1, or that ξ_3 follows immediately after ξ_2, etc. There may be intermediate derivatives. As for ζ we make no hypotheses concerning it. It may occur anywhere in the range.

We have assumed our system to be of the type described in the lemma. This means that if in the equation

$$\xi_i-\phi_i=0,$$

ϕ_i contains some derivative, say η anterior to ξ_i, then $\dfrac{\partial \phi_i}{\partial \eta}=0$ at the initial points $x_0, y_0, \ldots, \sigma_0, \ldots, \tau_0, \ldots$, of the solution.

Now let

$$\zeta=X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}.$$

Suppose $X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}$ to be absent from the group of first members of our system. We shall call it ζ_p. We are going to demonstrate that the system whose first members are

$$\xi_1, \xi_2, \ldots, \xi_{p-1}, \zeta,$$

can, after a linear homogeneous transformation of the independent variables,

$$x_1'=ax_1+bx_2+cx_3,$$
$$x_2'=a'x_1+b'x_2+c'x_3,$$
$$x_3'=a''x_1+b''x_2+c''x_3,$$

* *Annales de l'école normale supérieure, loc. cit.*, p. 426.

followed by a resolution with respect to the proper derivatives, be changed into a system whose first members are

$$\xi_1', \xi_2', \ldots, \xi_{p-1}', \xi_p'.$$

It will be sufficient to indicate the points wherein our proof differs from that of Delassus.

Let us return to the system S. It is our object to solve this system with respect to $\xi_1', \xi_2', \ldots, \xi_p'$, starting from the initial values $\xi_1^{(0)}, \xi_2^{(0)}, \ldots, \xi_{(p-1)}^{(0)}, \zeta^0, \eta^0, \ldots$, and we must show that the functional determinant does not vanish for these initial values. The functional determinant of our p equations, after they have been transformed with respect to $\xi_1', \xi_2', \ldots, \xi_p'$ is,

$$D=\left|\begin{array}{l} \dfrac{\partial\xi_1}{\partial\xi_1'}+\Sigma\dfrac{\partial\phi_1}{\partial\eta}\dfrac{\partial\eta}{\partial\xi_1'}, \ldots, \dfrac{\partial\xi_1}{\partial\xi_p'}+\Sigma\dfrac{\partial\phi_1}{\partial\eta}\dfrac{\partial\eta}{\partial\xi_p'}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \dfrac{\partial\xi_{p-1}}{\partial\xi_1'}+\Sigma\dfrac{\partial\phi_{p-1}}{\partial\eta}\dfrac{\partial\eta}{\partial\xi_1'}, \ldots, \dfrac{\partial\xi_{p-1}}{\partial\xi_p'}+\Sigma\dfrac{\partial\phi_{p-1}}{\partial\eta}\dfrac{\partial\eta}{\partial\xi_p'}, \\ \dfrac{\partial\zeta}{\partial\xi_1'}+\Sigma\dfrac{\partial\phi_p}{\partial\eta}\dfrac{\partial\eta}{\partial\xi_1'}, \ldots, \dfrac{\partial\zeta}{\partial\xi_p'}+\Sigma\dfrac{\partial\phi_p}{\partial\eta}\dfrac{\partial\eta}{\partial\xi_p'}. \end{array}\right|$$

If $\dfrac{\partial\phi_\lambda}{\partial\eta}\neq 0$ for the initial values, then by hypothesis $\dfrac{\partial\xi_\lambda}{\partial\xi_\lambda'}$ is anterior to η, $\{\lambda=1, 2, \ldots, p\}$.

And so following the same course of reasoning as Delassus, we retain only the first members of each element of the determinant. We are left with

$$\Delta=\frac{\partial(\xi_1, \xi_2, \ldots, \xi_{p-1}, \zeta)}{\partial(\xi_1', \xi_2', \ldots, \xi_p')}.$$

Then consider the minors of the above determinant $A_{p,p}, A_{p,p-1}, \ldots, A_{p,1}$. The diagonal of $A_{p,p}$ consists of the elements

$$\frac{\partial\xi_1}{\partial\xi_1'}, \frac{\partial\xi_2}{\partial\xi_2'}, \ldots, \frac{\partial\xi_{p-1}}{\partial\xi_{p-1}'}.$$

Each of these terms is of degree n in a, b', c''. For let $\xi_k'=X_1'^{a_1}X_2'^{a_2}X_3'^{a_3}$. Then, clearly, $\xi_k=X_1^{a_1}X_2^{a_2}X_3^{a_3}$, where $\{k=1, 2, \ldots, p-1\}$. After transforming we obtain

$$X_1^{a_1}X_2^{a_2}X_3^{a_3}=(aX_1'+a'X_2'+a''X_3')^{a_1}(bX_1'+b'X_2'+b''X_3')^{a_2}(cX_1'+c'X_2'+c''X_3')^{a_3}.$$

The coefficient of $X_1'^{a_1}X_2'^{a_2}X_3'^{a_3}$ is $a^{a_1}b'^{a_2}c''^{a_3}$. Hence $\dfrac{\partial\xi_k}{\partial\xi_k'}=a^{a_1}b'^{a_2}c''^{a_3}$, which contains no other quantities except a, b', c'', and is, consequently, of the n-th degree

in the above-mentioned quantities. So $A_{pp} \not\equiv 0$, but is of degree $n(p-1)$ in a, b', c''.

Let us now consider the expression $\dfrac{\partial \zeta_1}{\partial \xi'_p}$. Let $\zeta_1 = X_1^{a_1} X_2^{a_2} X_3^{a_3}$. Then,

$$\xi'_p = X_1^{a_1} X_2^{a_2-1} X_3^{a_3+1}, \quad \text{or} \quad X_1^{a_1-1} X_2^{a_2+1} X_3^{a_3}, \quad \text{or} \quad X_1^{a_1-1} X_2^{a_2} X_3^{a_3+1}.$$

Let us suppose that the first equality is the one which exists, for the reasoning will be the same in the other cases:

$$X_1^{a_1} X_2^{a_2} X_3^{a_3} = (aX'_1 + a'X'_2 + a''X'_3)^{a_1} (bX'_1 + b'X_2 + b''X_3)^{a_2} (cX'_1 + c'X'_2 + c''X'_3)^{a_3}.$$

The coefficient of $X_1^{a_1} X_2^{a_2-1} X_3^{a_3+1}$ is clearly obtained by expanding

$$a_1^{a_1} b'^{a_2-1} c''^{a_3} X_1'^{a_1} X_2^{a_2-1} X_3^{a_3} (bX'_1 + b'X_2 + b''X_3),$$

and is clearly $a^{a_1} b'^{a_2-1} b'' c''^{a_3}$ and is of degree $n-1$ in a, b', c''. Hence $\dfrac{\partial \zeta_1}{\partial \xi'_p}$ is of degree $n-1$ in the same quantities.

We can easily convince ourselves that $\dfrac{\partial \zeta_1}{\partial \xi'_1}, \dfrac{\partial \zeta_1}{\partial \xi'_2}, \ldots, \dfrac{\partial \zeta_1}{\partial \xi'_{p-1}}$ are of degree $\ngtr (n-1)$ in the a, b', c''. Also we easily demonstrate that all the minors $A_{p,\,p-1}, \ldots, A_{p,\,1}$ are of degree less than $n(p-1)$. Therefore, the whole determinant does not vanish identically, but is effectively of degree $n(p-1) + (n-1)$ in a, b', c''.

Thus we see that if $X_1^{a_1} X_2^{a_2} X_3^{a_3}$ is among the group of first members, either $X_1^{a_1} X_2^{a_2+1} X_3^{a_3}$ is a first member, or it can be made a first member by a linear homogeneous transformation followed by a resolution. Similar reasoning will show that we can make the same statement with respect to $X_1^{a_1+1} X_2^{a_2} X_3^{a_3-1}$ and $X_1^{a_1+1} X_2^{a_2-1} X_3^{a_3}$.

We can follow a course of reasoning parallel to that of Delassus in case the various equations of the system are not all of order n, or if there be more unknowns than one. It is easy to see that we can establish the convergence of the series formally satisfying our canonical system of equations by the method of Delassus, *i. e.*, by reducing the given system to a system of systems of the type of Kovalefski.

We should also note that a system can often be made canonical in more than one way. For example:

$$\frac{\partial^2 u}{\partial x_1^2} = f_1(x_1 x_2 x_3), \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = f_2 \quad \frac{\partial^2 u}{\partial x_1 \partial x_3} = c \frac{\partial^2 u}{\partial x_2^2},$$

and

$$\frac{\partial^2 u}{\partial x_1^2} = f_1(x_1 x_2 x_3), \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = f_2 \quad \frac{\partial^2 u}{\partial x_2} = \frac{1}{c} \frac{\partial^2 u}{\partial x_1 \partial x_3},$$

are equivalent forms, and both are canonical.

So far the author has shown, that since the canonical form developed is regular, it can be determined whether or not it is passive by a finite number of operations. But the knowledge that the canonical form is regular is not necessary for this purpose. In fact, it can readily be shown, that using the known theorems concerning the invariants λ and μ, we can determine whether or not a given system at certain points not solvable in orthonomic form, is passive throughout the entire space of n dimensions.